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Categories with Foundation

Context is everything

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Tiivistelmä — Referat — Abstract

Category theory is a field of mathematics that studies fields of mathematics. The concept of category formalizes the idea of a mathematical field of study. A category consists of objects and their composable interrelations. To study an object in an abstract category, one must study it through its surroundings.

The first chapter is about the set theoretic foundations of category theory. We start with recursion that gives us the capability to speak of first order predicate logic. The very basic overview of first order predicate logic is given to generally define a sufficiently big class of mathematical objects through the methods of logic. The Fundamental Theorem of Model Morphism is proven, which is a considerable generalization of The Fundamental Theorem of Homomorphism of group theory. After predicate logic we move to study the basics of set theory. We talk about ordinals, which allows us to add a new axiom to set theory that make it easier to work with categories. The Axiom of Universes is assumed and the needed theory is constructed to understand inaccessible cardinals.

Second chapter introduces category theory in relation to other mathematical fields. In the first chapter we look at objects, morphisms, subobjects, products and exponentials in a category. These concepts are ubiquitous in mathematics and happen to be present themselves heavily in category theory itself. An important topic in the presented thesis is the many different ways morphisms converse about structure and how one classifies these, for example, as isomorphisms or embeddings. The concept of an embedding and an identification are defined in a concrete category. The concept of inductance and coinductance of structure is defined, which leads the conversation to topological categories, which we use to show that the category of topological spaces is complete and cocomplete.

The third chapter uses these tools created in an arbitrary category and applies them to the category of small categories. The talk about foundations makes it possible to consider any meta category as a small category. Third chapter tackles the concept of functors, subcategories, quotient categories and exponentials of categories. This opens the door to talk about natural transformations, which will be the canonical morphisms between functors and make it possible to define adjoint functors.

In the third chapter an isomorphism between functors is defined. Fourth chapter applies this knowledge in the study functors that are isomorphic to hom-functors and these are called representable functors. The study of representable functors yields an important lemma of category theory called Yoneda lemma. Yoneda lemma implies the fully faithfulness of Yoneda embedding: If two objects are contextually equivalent, they are equivalent. Yoneda lemma allows us to characterize some constructions with a single object.

The fifth chapter studies how to define new and interesting objects through limits and colimits. Similarly as in the theory of metric spaces one is interested in completeness of the spaces. The way to study completeness in the metric setting is to look at the behaviour of limits. We do the same with categories. In the fifth chapter we find out how limit procedures preserve in constructions and how they behave when functors pass them forward.

The last and the sixth chapter concentrates on adjoint functors. The general and special adjoint theorems due to Peter Freyd are proven. Using the General Adjoint Functor Theorem, we prove the existence of a left adjoint functor for all suitable forgetful functors among algebraic categories.

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Abstract

We develop the theory of categories from foundations up. The thesis culminates in theorem in which we assert that any concrete functor between categories of models of algebraic theories, where the codomain categories alphabet does not contain relational information, has a left adjoint functor. This theorem is based on The General Adjoint Functor Theorem by Peter Freyd.

The first chapter is about the set theoretic foundations of category theory. We present the needed ideas about recursion so that we may define what is meant by first order predicate logic. The first chapter ends in the exposition of the connection between the Grothendieck universes and the inaccessible cardinals. The second chapter starts our conversation about categories and functors between categories. We define properties of morphisms, subobjects, quotient objects and Cartesian closed categories. Furthermore, we talk about embedding and identification morphisms of concrete categories.

Much of the third chapter is to show that the category of small categories is a Cartesian closed category. This leads us to talk about natural transformation and canonical constructions relating to functors. To define equivalences and their generalizations, adjoint functors, natural transformations are needed. The fourth chapter enlarges our knowledge about hom-functors and their adjacent functors, representable functors. The study of representable functors yields a profound lemma called Yoneda lemma. Yoneda lemma implies the fully faithfulness of Yoneda embedding.

The fifth chapter concentrates to limit operations in a category, which leads us to talk about completeness. We find out how limit procedures are preserved in constructions and how they behave when functors pass them forward. The last chapter is about adjoint functors. The general and the special adjoint functor theorems, due to Peter Freyd, are proven. Using The General Adjoint Functor Theorem, we prove the existence of a left adjoint functor for all suitable forgetful functors among algebraic categories.

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> "Reflect upon your present blessings, of which every man has plenty; not on your past misfortunes, of which all men have some." —Charles Dickens

Contents

Introduction

From any set and any collection of algebraic axioms, we produce a method that constructs a free and canonical algebraic structure with respect to the axioms and the set. A singleton set and the axioms of monoid produce the monoid of natural numbers. A singleton set associated with the group axioms produce the group of integers. More generally even, the method shows that any monoid freely creates a group. Formed more exactly, the most important result in this thesis is Theorem 6.5, which says that every continuous concrete functor between model categories of first order theories, where the domain category is algebraic and the functor preserves embeddings, is a right adjoint functor.

We develop the theory from foundations up. Before tackling the definitions of first order predicate logic, we familiarize ourselves with recursion. The notion of injective-recursive construction of collections is of fundamental importance, since it permits the existence of important recursively defined functions. For example, Theorem 1.9 is needed to satisfactorily give Tarski's Definition of Truth.

We classify first order formulas by which connectives are used in their creation. It so happens that a surjective model morphism and a globally full injective model morphism preserve and reflect the truth of positive and universal formulas between models, respectively. This is proved in Theorem 1.41, which is of importance in showing the completeness of category of models satisfying an algebraic theory.

We see that Zermelo-Fränkel set theory is a first order theory that is almost rich enough in structure to encapsulate our conversation about categories. We develop the basic theory of ordinals and cardinals to appreciate the Axiom of Universes, which we choose to assume. The Axiom of Universes is equivalent with the existence of a hierarchy of Grothendieck universes, which encompass all sets. We find a bijective correspondence with inaccessible cardinals and Grothendieck universes, which model set theory. We assume from our universe of sets that the class of inaccessible cardinals are unbounded in the class of ordinals. This assumption is called the Axiom of Universes. With it we justify categorical constructions later on. The category of sets is defined with a Grothendieck universe in mind. The category of sets never encompass all sets, just the sets in some Grothendieck universe.

The second chapter focuses on the basics of category theory. We present categories from the perspective of graph theory, since category can be defined as a compositional multigraph. We classify different types of morphisms. Most important morphism classes are isomorphisms, monomorphisms and epimorphisms. The latter two are generalizations of injective and surjective functions. The notion of inducing topological structure via a collection of functions generalizes to categorical inductance. The inductance of structure leads us to define topological functors, embeddings and identifications. In Example 2.43 we show that in a category of models which satisfy a positive theory, embedding is characterized as a globally full injection. Similarly the concept of globally full surjective model morphism characterizes the concept of identification in a positive category. The latter part of the second chapter looks what structure hides inside an object and we start to consider multiplying, summing and exponentiating objects. We define Cartesian closed categories and show that categorical composition can be internalized in Cartesian closed categories.

The third chapter applies the general theory of categories to one specific category; namely the meta category of all categories. By suspecting that the category of small categories is Cartesian closed we find the concept of natural transformation. From this we prove that the category of small categories is Cartesian closed. Natural transformations make it possible to define equivalence of categories and their generalizations, adjoint functors, which are closely related to the concept of a basis. The Adjoint Creation Lemma 3.39 makes this connection to the concept of basis explicit.

The fourth chapter deepens the understanding of functors. Yoneda lemma makes it possible to characterize a large collection of set valued functors by a single suitable object. These objects are called representations of functors. Yoneda lemma also yields a commonly used proof technique by category theorists: An object is characterized up to an isomorphism by how morhpisms leave it and by how morphisms arrive to it. Yoneda lemma helps us to understand adjuncts of adjoint functors.

The fifth chapter explores the existence of limits and colimits in categories and how these properties are inherited via different constructions. When they exists, limits and their dual colimits, are universal objects defined by diagrams. Completeness of a category permits important constructions. The classical adjoint functor theorems, which are proven in the sixth chapter, heavily rely on the completeness assumptions. Every functor defines a cone and a cocone functor. The Cone Functor Theorem 5.19 shows how some properties of cone functors are inherited from the underlying functor. Perhaps in an original way, we show that right adjoint functors preserve limits, which is proven in Theorem 5.21.

The last and the sixth chapter concentrates on two classical adjoint functor theorems, the General Adjoint Functor and the Specials Adjoint Functor theorems. With the General Adjoint Functor theorem we prove Theorem 6.6, which says informally that algebraic models, without relational information, freely generate models that satisfy more algebraic axioms. To satisfy the authors intrigue, we give a general condition when an algebraic model category contains a tensor product.

Chapter 1

Foundations for categories

There's an interesting category called **Set**, the category of sets. As one can imagine, the category **Set** is quite a large object of study. To have a formalized theory of big objects, one needs to enlarge the formal mathematical language. There are multiple routes to give foundations. One is to recognize that the collection of categories has a categorical structure. One can then try to give axioms for the meta category of all categories, as one does with set theory. This leads into a deep theory of infinity categories. Easier route is to have a formal class theory. Von Neumann-Barneys-Gödel set theory (**NBG**) creates a language to talk about bigger objects than sets. This is a nice theory, because, as Michael Shulman puts it in his text "Set theory for category theory"[7], **NBG** has no ontological commitment: Theorems proved about sets in **NBG** can be proved in Zermelo-Fränkel set theory with the axiom of choice (**ZFC**). The problem is that just adding one new type of big sets doesn't do enough, since we will talk about 'the category' of all categories.

We shall choose to use Grothendieck universes to do category theory sets that model set theory. We will assume a weak large cardinal axiom that gives us an infinite hierarchy of sets called Grothendieck universes. Each Grothendieck universe defines a model for **ZFC**.

Notation

We use three different terms; sets, classes and collections. Sets and classes are special kinds of collections. We make this differentiation because later on we give formal definitions how sets behave. The axioms of set theory, for example, imply that all sets form a hierarchy that can be constructed from the empty set. We use the word 'collection' in a colloquial sense and not in a formal sense. A class is a special kind of collection of sets that are formally defined through a formula that identifies a collection of sets. Every set is a class, but for example the class of all sets is not a set.

Next we code some concepts in to the language of sets.

Definition 1.1. We give the following definitions:

1. Let X be a set. We define $\mathcal{P}(X)$ to be the set of all subsets of X.

2. If X and Y are sets, we define the ordered pair (X, Y) to be the set $\{\{X\}, \{X, Y\}\}$. If X_i is a set for $i \leq n+1$, $n \in \mathbb{N}$, then $(X_0, \ldots, X_n, X_{n+1}) := ((X_0, \ldots, X_n), X_{n+1})$.

3. Let $X_1, \ldots, X_n$ be sets for $n \geq 2$. We define the Cartesian product $X_1 \times \ldots \times X_n$ to be the set of ordered pairs $(x_1, \ldots, x_n)$ where $x_i \in X_i$ for $i \leq n$. If $X_i = X$ for all $i \leq n$, then we denote the set $X_1 \times \ldots \times X_n$ by X^n. We define $X^0 := \{\emptyset\}$ and $X^1 = X$.

4. Let X_1, X_2 be sets. We define the disjoint union $X_1 \sqcup X_2$ to be the set $\{1\} \times X_1 \cup \{2\} \times X_2$. We may then identify the set X_i with the set $\{i\} \times X_i$ for $i = 1, 2$.

5. We call a set R a relation, if there exist sets X and Y where $R \subset X \times Y$. A tuple (R, X, Y) is then said to be a relation from X to Y. For a relation R we define the opposite relation $R^{-1} := \{(y, x) \mid (x, y) \in R\}$. The domain of definition of R is the set

$$\{x \mid (x, y) \in R \text{ for some } y\},$$

denoted $\mathrm{Dom}(R)$. We define the image of a set A under the relation R as

$$R[A] := \{y \mid (x,y) \in R \text{ for some } x \in A\}.$$

The image of a relation R is the image of the domain of R and is denoted by $\mathrm{im}(R)$. The pre-image of A under the relation R is $R^{-1}[A]$.

6. If R and S are relations, then we define the composition

$$S \circ R := \{(x,z) \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y\}$$

7. Let R be a relation. Then R is said to be a function, if for every element x, y and y' holds that $(x,y),(x,y') \in R$ implies $y = y'$. The element y is then denoted as $R(x)$ and we say that x maps to y under R and denote $x \overset{R}{\mapsto} y$. If furthermore, (R, X, Y) is a relation from X to Y, the tuple (R, X, Y) is called a partial function and denoted $R : X \rightharpoondown Y$. If R is a partial function from X to Y and the domain of R is equal to X, then (R, X, Y) is called a function from X to Y and denoted $R : X \to Y$.

8. If R is a relation and A and B are sets, we say that $R \cap (A \times B)$ is a restriction of R and denote $R \mid$, given that the sets A and B are clear from context. We will denote $R \mid A$ to mean $R \cap (A \times \mathrm{im}(R))$

9. Let $f_i : X_i \to Y_i$ be a function for $i = 1, 2$. We define $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ to be the function $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$. If $X := X_1 = X_2$, then we will denote with (f_1, f_2) the function $X \to Y_1 \times Y_2$, where $x \mapsto (f_1(x), f_2(x))$.

10. Let X, Y and Z be sets. We define Y^X and $[X, Y]$ to be the set of all functions f where $f : X \to Y$, which is also denoted by $[X, Y]$. Assume that $f \subset X \times Y$ is a relation, we define $f_* : \mathcal{P}(Z \times X) \to \mathcal{P}(Z \times Y)$, $g \mapsto f \circ g$ and, if f is a function, then this restricts to a map $[Z, X] \to [Z, Y]$.

11. If X_i and I are sets for $i \in I$, then we denote

$$\bigsqcup_{i \in I} X_i = \cup_{i \in I} \{i\} \times X_i \text{ and}$$
$$\prod_{i \in I} X_i = \{f : I \to \bigsqcup_{i \in I} X_i \mid f(i) \in X_i\}.$$

We call the former set a disjoint union of sets $X_i, i \in I$ and the latter a product of sets $X_i, i \in I$. Axiom of choice implies that, if the sets X_i are non-empty, then $\prod_{i \in I} X_i$ is non-empty. We may denote $x \in \prod_{i \in I} X_i$ by $(x_i)_{i \in I}$ where $x_i = x(i)$. We identify $X_1 \times \ldots \times X_n$ with $\prod_{i \in I} X_i$, where $I = \{1, \ldots, n\}$.

The composition of relations is associative: Let P, R, S be relations. Now $(S \circ R) \circ P = S \circ (R \circ P)$. There are noteworthy functions: Every set X has an identity function associated to it denoted by $id_X : X \to X$ where $x \mapsto x$. If $A \subset X$, then the restriction of the identity to the map $A \to X$ is denoted $A \hookrightarrow X$ and the restriction map is called an inclusion.

1.1 Recursion

We prove that we may give recursive definitions for collections. We choose to restrict ourselves to handle collections similarly as sets, but without the axiom of choice. For example we assume the extensionality for collections: Two collections are equal if and only if they share the same elements. Later we apply these results for sets. Additionally, we don't want stumble upon paradoxes.

Theorem 1.2 (The Fundamental Theorem of Recursion)**.** *Let X and I be collections. Let $f_i : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function, where $f_i(A) \subset f_i(B)$ for $A \subset B \subset X$, for $i \in I$. Denote*

$$\mathcal{H} := \{A \subset X \mid f_i(A) \subset A \textit{ for all } i \in I\}$$

and let $\mathcal{A} \subset \mathcal{H}$ be non-empty. Then the intersection of collections in $\mathcal{A}$, denoted $\cap \mathcal{A}$, is an element of $\mathcal{H}$. Especially there exists the smallest collection $A \subset X$, where $f(A) \subset A$, with respect to inclusion.

Proof. Denote with B the intersection of elements in $\mathcal{A}$. It suffices to show that $B \in \mathcal{H}$. Let $A \in \mathcal{A}$. Now $B \subset A$ and so

$$f_i(B) \subset f_i(A) \subset A$$

for all $A \in \mathcal{H}$ and hence $f_i(B) \subset B$ for all $i \in I$. Therefore $B \in \mathcal{H}$. Choosing $\mathcal{A} = \mathcal{H}$, which is non-empty, since $X \in \mathcal{H}$, we attain the smallest element in $\mathcal{H}$ with respect to inclusion. □

Definition 1.3. Let X and I be collections and let $f_i : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function for $i \in I$. Assume that $f_i(A) \subset f_i(B)$ for $A \subset B \subset X$ and $i \in I$. We denote by $\mathrm{cl}(f_i)_{i\in I}$ the smallest collection A where $f_i(A) \subset A$ for all $i \in I$. The set $\mathrm{cl}(f_i)_{i\in I}$ is said to be the recursive closure of closure operations $f_i, i \in I$. In practice we may leave the operations f_i implicit and call the collection $A := \mathrm{cl}(f_i)_{i\in I}$ the recursive closure of closure operations $f_i(A) \subset A$.

Let $B \subset X$. The proof technique where we deduce that $A \subset B$ by showing $f_i(B) \subset B$ for all $i \in I$ is called induction on the structure of A with respect to the closure operations $f_i, i \in I$. If the closure operations are clear from context, we may not mention them.

Let X and α_i be collections for $i \in I$, where I is an index collection. Assume that $R_i \subset X^{\alpha_i} \times X$ for all $i \in I$. Then by the Fundamental Theorem of Recursion there exists the smallest collection $A \subset X$, where

$$R_i[A^{\alpha_i}] \subset A$$

for all $i \in I$. We may also call the relations R_i for $i \in I$ closure operations on X and denote the collection A with $\mathrm{cl}(R_i)_{i\in I}$. If $\alpha_i = \emptyset$, then the relation $R_i \subset X^{\alpha_i} \times X$ can be identified as a subcollection $R_i[\emptyset]$ of X and the collection A is called the closure of $R_i[\emptyset]$ with respect to the closure operations $R_j, j \neq i$. The following examples will shed light to this.

Example 1.4.

1. Let G be a group with e_G as the neutral element and $A \subset G$. Then there exists the smallest subgroup H of G containing A. This is seen directly by choosing

$$R_i = \begin{cases} \{e_G\} \cup A, \text{ if } i = 0 \\ (g, g') \mapsto gg', \text{ if } i = 1 \\ g \mapsto g^{-1}, \text{ if } i = 2. \end{cases}$$

   Now the smallest collection H of the form B, where $R_i[B^{n_i}] \subset B, i = 0, 1, 2$, is exactly the smallest subgroup of G containing A. Other way to say the same is that $H \subset G$ is the closure of $A \cup \{e_G\}$ with respect to the closure operations

$$\begin{cases} ab \in H \text{ for } a, b \in H \text{ and} \\ b^{-1} \in H \text{ for } b \in H. \end{cases}$$

2. Let (X, τ) be a topological space. We define the collection of Borel sets of (X, τ) to be the set $\mathrm{Bor}(X, \tau) \subset \mathcal{P}(X)$ that is recursively defined by the closure operations

$$\begin{gathered} V \in \mathrm{Bor}(X, \tau), \text{ for } V \in \tau, \\ X \setminus A \in \mathrm{Bor}(X, \tau), \text{ for } A \in \mathrm{Bor}(X, \tau) \text{ and} \\ \cup_{i\in\mathbb{N}} A_i \in \mathrm{Bor}(X, \tau), \text{ for } A_i \in \mathrm{Bor}(X, \tau), i \in \mathbb{N}. \end{gathered}$$

Notice that, if $A \subset X, I, \alpha_i$ are collections and if $f_i : X^{\alpha_i} \rightharpoonup X$ is a partial functions for $i \in I$, then any element in $b \in B := \mathrm{cl}(A, f_i)_{i\in I}$ is either an element in A or there exist some $i \in I$ and $b' \in B^{\alpha_i} \cap \mathrm{Dom}(f_i)$ such that $f_i(b') = b$. This is seen directly by an induction on the structure of B.

Definition 1.5. Assume that X, I, J and α_i are collections for all $i \in I$. Assume that $f_i : X^{\alpha_i} \rightharpoonup X$ is a partial function for $i \in I$ and let $A_j \subset X$ for $j \in J$. Denote $A = (A_j)_{j\in J}$ and $f = (f_i)_{i\in I}$. We say that the collection (A, f) is jointly injective, if for every $b \in B := \mathrm{cl}(A, f)$ holds one and only one of the following conditions:

$$\begin{aligned} &b \in A_j \text{ for some unique } j \in J \\ &b = f_i(b') \text{ for a unique } i \in I \text{ and a unique } b' \in B \cap \mathrm{Dom}(f_i). \end{aligned}$$

Additionally, we say that B is injective-recursively defined by collection (A, f).

Notice that the collection of natural numbers $\mathbb{N}$ is injective-recursively defined by $(\{0\}, S)$ where $S : \mathbb{N} \to \mathbb{N}$ is the follower function $n \mapsto n + 1$. The injective-recursiveness of $(\{0\}, S)$ is actually equivalent with the set theoretic formulation of natural numbers.

Next we show a method to create injectively-recursive collections: Let Σ be a be collection of symbols that contain the left bracket $'('$, the right bracket $')'$ and the comma $','$. Define the collection Σ^+ to be the collection of all finite sequences of symbols in Σ. The collection Σ^+ is called the free monoid over Σ, since the monoid structure is defined by concatenation, gluing of sequences together. We identify $x = (x_1, \dots, x_n) \in \Sigma^+$ with $x_1 \dots x_n$ for symbols $x_i \in \Sigma, i \leq n$ and especially we identify the elements of Σ with the singleton sequences of Σ^+. We say that x is a word of symbols in Σ of length n. If $0 < k \leq n$, then we say that $w = x_1 \dots x_k$ is an initial segment of x, denoted $w \prec x$. The word w is a strict initial segment of x, if $k < n$.

Let $A, E, F, R \subset \Sigma^+$ be disjoint collections of non-empty sequences. Denote $L = E \cup F \cup R$ and let $l : L \to \mathbb{N}_1$ be a function, where $l(f) = 2$ for $f \in F$.[1] Define functions $Q_x : (\Sigma^+)^{l(x)} \to \Sigma^+$ for $x \in L$, where

$$\begin{aligned} Q_e(a_1, \dots, a_{l(e)}) &= e a_1 \dots a_{l(e)} \\ Q_f(a_1, a_2) &= (a_1 f a_2) \\ Q_r(a_1, \dots, a_{l(r)}) &= r(a_1, \dots, a_{l(r)}) \end{aligned}$$

for $e \in E, f \in F, r \in R$ and $a_i \in \Sigma^+, i \in \mathbb{N}$. We call an element $x \in E \cup F \cup R$ a connective of pair (x, Q_x) and the function Q_x an application of the connective x of the pair (x, Q_x). Let B be the collection recursively defined by the collection $(A, (Q_x)_{x \in L})$.

Theorem 1.6. *Let $A, E, F, R \subset \Sigma^+$ be disjoint collections of non-empty sequences and denote $L = E \cup F \cup R$. Let $l : L \to \mathbb{N}_1$ and $Q_x : (\Sigma^+)^{l(x)} \to \Sigma^+$ be functions for $x \in L$ and denote $B = cl(A, (Q_x)_{x \in L})$ similarly as above. Assume the following statements:*

- *Let $x, y \in A \cup E \cup R$. Then the first symbol of x is not the left bracket, and if $x \prec y$, then $x = y$.*
- *Let $f, f' \in F$. If $f \prec f'$, then $f = f'$.*

Then the following claims hold:

1. *Let $w, b \in B$. Assume that $w \prec b$. Then $w = b$.*
2. *The collection B is injective-recursively defined by $(A, (Q_x)_{x \in L})$.*

Proof.

1. Before we begin notice the following fact: Let $a, b, c, d \in \Sigma^+$ be non-empty sequences. If $ab \prec cd$, then $a \prec c$ or $c \prec a$. Let I contain exactly those elements $b \in B$ where the following conditions hold:

   If $w \prec b$ and $w \in B$, then $w = b$.
   If $w_1 \in B, w_2 \in \Sigma^+$ and $b \prec w_1 w_2$, then $b = w_1$.

   We will show that $I = B$. First we'll show that $A \subset I$. Let $a \in A$. Assume that $w \in B$ and $w \prec a$. Since the word w cannot start with an element of E or R nor can it have the left bracket as the first symbol, it follows that $w \in A$. Thus $w = a$. Assume then that $a \prec w_1 w_2$ where $w_1 \in B$ and $w_2 \in \Sigma^+$. So $w_1 \in A$ and thus $w_1 = a$. Hence $A \subset I$.

   Let $e \in E, f \in F$ and $r \in R$ and denote $k = l(e)$ and $n = l(r)$. Let $a_i \in I$ for $i \in \mathbb{N}$. It suffices to show that

$$\begin{aligned} e a_1 \dots a_k &\in I, \\ (a_1 f a_2) &\in I \text{ and} \\ r(a_1, \dots, a_n) &\in I. \end{aligned}$$

[1] By $\mathbb{N}_1$, we denote the collection of natural numbers starting from 1.

Let $w \prec ea_1 \dots a_k$ and $w \in B$. Thus $w = eb_1 \dots b_k$ for some $b_1, \dots b_k \in B$. Since $a_1 \in I$, it follows that $a_1 = b_1$. By continuing inductively, we attain $b_i = a_i$ for $i \leq k$. Hence $w = ea_1 \dots a_k$. Assume then that $ea_1 \dots a_k \prec w_1 w_2$ for $w_1 \in B$ and $w_2 \in \Sigma^+$. Thus $w_1 = eb_1 \dots b_k$ for some $b_1, \dots, b_k \in B$. Now again $a_i = b_i$ for all $i \leq k$ and hence $ea_1 \dots a_n = w_1$. So $ea_1 \dots a_n \in I$

Let $w \prec (a_1 f a_2)$ and $w \in B$. Now $w = (b_1 f' b_2)$ for some $f' \in F$ and $b_1, b_2 \in B$. Now either $a_1 \prec b_1$ or $b_1 \prec a_1$ and hence $a_1 = b_1$. Thus $f' = f$ and $b_2 = a_2$. Therefore $w = (a_1 f a_2)$. Assume then that $(a_1 f a_2) \prec w_1 w_2$ where $w_1 \in B$ and $w_2 \in \Sigma^+$. Now $w_1 = (b_1 f' b_2)$ for some $b_1, b_2 \in B$ and $f' \in F$. Therefore $a_1 \prec b_1 f' b_2) w_2$. Thus $a_1 = b_1, f = f'$. Now $a_2 \prec b_2) w_2$ and so we have $a_2 = b_2$. Therefore $(a_1 f a_2) \in I$.

Similarly we obtain $r(a_1, \dots, a_n) \in I$. Hence $I = B$.

2. Let $b \in B$. We need to show that one and only one of the following conditions hold:

$$\begin{aligned} &b \in A, \\ &b = Q_x(a_1 \dots a_{l(x)}) \text{ for a unique } x \in L \text{ and unique } a_1 \dots a_{l(x)} \in B. \end{aligned}$$

This follows in a similar manner as part 1.

□

Theorem 1.7 (Recursion on a Recursive Closure)**.** *Let X, Y, I and α_i be collections for $i \in I$. Assume that $A \subset X$, $f_i : X^{\alpha_i} \rightharpoonup X$ is a partial function for $i \in I$. Assume that $B \subset X$ is injective-recursively defined by the collection $(A, f_i)_{i \in I}$. Assume that $g_i : Y^{\alpha_i} \to Y$ is a function for $i \in I$ and $s : A \to Y$. Then there exists a unique extension $S : B \to Y$ of s, where*

$$\begin{aligned} S(a) &= s(a) \textit{ for } a \in A \textit{ and} \\ S(f_i(b)) &= g_i(S_*(b))) \textit{ for } b \in \mathit{Dom}(f_i) \cap B^{\alpha_i}, i \in I. \end{aligned}$$

In other words the diagram

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr, "s"'] & B \arrow[d, dashed, "S"] & B^{\alpha_i} \cap \mathit{Dom}(f_i) \arrow[l, "f_i|"'] \arrow[d, dashed, "S_*|"] \\ & Y & Y^{\alpha_i} \arrow[l, "g_i"] \end{tikzcd}$$

commutes for unique function S and for every $i \in I$.[2] *Furthermore, the image of S is a subset of $cl(s[A], g_i)_{i \in I}$.*

Proof. Let $s : A \to Y$. First we show the uniqueness. Assume that there are two functions $S, S' : B \to Y$ that satisfy the claim. Let $J = \{b \in B \mid S(b) = S(b')\}$. Clearly $A \subset J$. We will show that $f_i[J^{\alpha_i}] \subset J$ for all $i \in I$. Let $b \in J^{\alpha_i} \cap \mathrm{Dom}(f_i)$. Now

$$S(f_i(b)) = g_i(S_*(b)) = g_i(S'_*(b)) = S'(f_i(b)).$$

Therefore $f_i(b) \in J$. Hence $J = B$ and thus the uniqueness is seen.

To show the existence, define $S \subset X \times Y$ to be the collection recursively defined by the closure operations

$$(a, s(a)) \in S \text{ for } a \in A \text{ and} \tag{a}$$
$$(f_i(b), g_i(y)) \in S \text{ for } i \in I, b \in \mathrm{Dom}(f_i) \text{ and } y \subset S \circ b, \text{ where } y \in X^{\alpha_i}. \tag{b}$$

Notice that $S \subset B \times \mathrm{cl}(s[A], g_i)_{i \in I}$, because the latter relation also satisfies the closure conditions (a) and (b). Next we will show that S defines a function by induction on the structure of B. Set

$$J := \{b \in B \mid \text{there exists a unique } y \in Y \text{ where } (b, y) \in S\}.$$

We show that $A \subset J$. Let $a \in A$. Now by definition of S, $(a, s(a)) \in S$. Assume that $(a, y) \in S$ for some $y \neq s(a)$. Now $S \setminus \{(a, y)\}$ satisfies the closure condition (a). Since no element of the form

[2] We call such a drawing a diagram and, if given any paths between two vertices in the diagram yield the same function with composition, we say that the diagram is commutative.

$(f_i(b), y')$, where $b \in B$ and $y \in Y$, is taken away from S, we see that $S \setminus \{(a, s(a))\}$ satisfies the closure condition (b). This is a contradiction and so $a \in J$.

We need to show that $f_i[J^{\alpha_i}] \subset J$ for every $i \in I$. Let $i \in I$ and $b \in J^{\alpha_i} \cap \mathrm{Dom}(f_i)$. We are required to show that $f_i(b) \in J$. Now $b = (b_k)_{k \in \alpha_i}$ for some $b_k \in J, k \in \alpha_i$. Thus there exists unique $y_k \in Y$, where $(b_k, y_k) \in S$ for every $k \in \alpha_i$. Denote $y = (y_k)_{k \in \alpha_i}$. Now $y = S \circ b$ and $y \in X^{\alpha_i}$. By the definition of S, $(f_i(b), g_i(y)) \in S$.

To show uniqueness, assume for the contrary that $(f_i(b), y') \in S$ for some $y' \neq g(y)$. Yet again consider $S' := S \setminus \{(f_i(b), y')\}$. The relation S' satisfies the closure condition collection (a). Next we show that the closure condition (b) holds as well. Assume that $j \in I, c \in \mathrm{Dom}(f_j)$ and $z \subset S' \circ c$, where $z \in X^{\alpha_j}$. Now $z \subset S \circ b'$ and hence by the definition of S, $(f_j(c), g_j(z)) \in S$. It suffices to show that $(f_j(c), g_j(z)) \in S'$. Assume for a contradiction that $(f_j(c), g_j(z)) \notin S'$. Therefore $f_j(c) = f_i(b)$ and $g_j(z) = y'$. By assumption $i = j$ and $c = b$. Now $y = S \circ b$ and $z \subset y$. Since $z, y \in X^{\alpha_i}$ and $z \subset y$, $z = y$. This leads to a contradiction that $y' = g_j(z) = g_i(y)$. Thus $(f_j(c), g_j(z)) \in S'$ and therefore S' satisfies the recursion condition (b). This again leads to a contradiction, because $S' \subsetneq S$. Thus $y' = g_i(y)$ and therefore $f_i(b) \in J$. Hence $J = B$.

Now we have seen that the relation S defines a function, similarly denoted, $S : B \to Y$. Directly from the closure conditions of relation S, we attain $S(a) = s(a)$ for all $a \in A$ and $S(f_i(b)) = g_i(S_*(b))$ for all $b \in \mathrm{Dom}(f_i) \cap B$ and $i \in I$. $\square$

Example 1.8. Let S be the follower function of natural numbers $\mathbb{N}$. Let X be a set, $a \in X$ and let $f : X \to X$ be a function. By Theorem 1.7 there exists a unique map $g : \mathbb{N} \to X$ where $g(0) = a$ and $g(S(n)) = f(g(n))$. Notice that $g = (a, f(a), f(f(a)), \ldots)$.

Theorem 1.9. *Let X, I, α_i, Y and V be collections for all $i \in I$. Let $f_i : X^{\alpha_i} \to X$ be functions and $A \subset X$. Assume that B is injective-recursively defined by the collection $(A, f_i)_{i \in I}$. Assume still that $g_i : V \times X^{\alpha_i} \times Y^{\alpha_i} \to Y$ and $s : V \times A \to Y$ are functions for $i \in I$. Then there exists a unique function $S : V \times B \to Y$ where the diagram*

$$\begin{tikzcd}
V \times A \arrow[r, hook] \arrow[dr, "s"'] & V \times B \arrow[d, dashed, "S"] & V \times B^{\alpha_i} \arrow[l, "id \times f_i|"'] \arrow[d, dashed, "(j, S_\circ|)"] \\
 & Y & V \times X^{\alpha_i} \times Y^{\alpha_i} \arrow[l, "g_i"]
\end{tikzcd}$$

commutes and the function j is an inclusion $V \times B^{\alpha_i} \hookrightarrow V \times X^{\alpha_i}$, for every $i \in I$.[3]

Proof. We may assume that $B = X$, since otherwise we may restrict f_i to B^{α_i} for $i \in I$. By a simple induction on the structure of B we attain the uniqueness.

To show the existence, we first assume that V is just a singleton collection and hence disappears from the diagram above. In this special case we are looking for a function S such that the diagram

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s"'] & X \arrow[d, dashed, "S"] & X^{\alpha_i} \arrow[l, "f_i"'] \arrow[d, dashed, "(id, S_*)"] \\
 & Y & X^{\alpha_i} \times Y^{\alpha_i} \arrow[l, "g_i"]
\end{tikzcd}$$

commutes. We define

$$t_j : (X \times Y)^{\alpha_j} \to X^{\alpha_j} \times Y^{\alpha_j}, (x_i, y_i)_{i \in \alpha_j} \xmapsto{t_j} ((x_i)_{i \in \alpha_j}, (y_i)_{i \in \alpha_j})$$

$$g'_j : (X \times Y)^{\alpha_j} \to X \times Y, (x_i, y_i)_{i \in \alpha_j} \xmapsto{g'_j} (f_j((x_i)_{i \in \alpha_j}), g_j((x_i)_{i \in \alpha_j}, (y_i)_{i \in \alpha_j}))$$

$$s' : A \to X \times Y, a \xmapsto{s'} (a, s(a))$$

for all $j \in I, a \in A, y \in Y$ and $(x_i, y_i)_{i \in I} \in (X \times Y)^{\alpha_i}$. By Theorem 1.7 there exists a function $S' : X \to X \times Y$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s'"'] & X \arrow[d, dashed, "S'"] & X^{\alpha_i} \arrow[l, "f_i"'] \arrow[d, dashed, "S'_*"] \\
 & X \times Y & (X \times Y)^{\alpha_i} \arrow[l, "g'_i"]
\end{tikzcd}$$

[3]We define that $S_\circ : V \times X^{\alpha_i} \to Y^{\alpha_i}, (v, x) \mapsto (S(v, x_k))_{k \in \alpha_i}$ for $i \in I$.

commutes. Define maps $p : X \times Y \to X, q : X \times Y \to Y$ and , where $p(x, y) = x, q(x, y) = y$ and for all $x \in X, y \in Y$. Now both of the diagrams

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s'"'] \arrow[ddr, bend right, hook] & X \arrow[d, dashed, "S'"] & X^{\alpha_i} \arrow[l, "f_i"'] \arrow[d, dashed, "S'_*"] \\
& X \times Y \arrow[d, dashed, "p"] & (X \times Y)^{\alpha_i} \arrow[l, "g'_i"'] \arrow[d, dashed, "p_*"] \\
& X & X^{\alpha_i} \arrow[l, "f_i"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s'"'] \arrow[ddr, bend right, "s"'] & X \arrow[d, dashed, "S'"] & X^{\alpha_i} \arrow[l, "f_i"'] \arrow[d, dashed, "S'_*"] \\
& X \times Y \arrow[d, dashed, "q"] & (X \times Y)^{\alpha_i} \arrow[l, "g'_i"] \arrow[d, dashed, "t_i"] \\
& Y & X^{\alpha_i} \times Y^{\alpha_i} \arrow[l, "g_i"]
\end{tikzcd}
$$

are commutative. Since the left diagram commutes, we attain $p \circ S' = id_X$. Because $S' = (pS', qS') = (id_X, qS')$, it holds that qS' is the function we are looking for.

Now for the general case. First we define functions $s' : A \to Y^V$ and $h_i : X^{\alpha_i} \times (Y^V)^{\alpha_i} \to Y^V$, where

$$
\begin{gathered}
s'(a)(v) = s(a, v) \\
h_i(x, (t_j)_{j \in \alpha_i})(v) = g_i(v, x, (t_j(v))_{j \in \alpha_i})
\end{gathered}
$$

for $v \in V, y \in Y, x \in X^{\alpha_i}$ and $(t_j)_{j \in \alpha_i} \in (Y^V)^{\alpha_i}$. By the previous, we have a unique function $S' : X \to Y^V$ where the diagram

$$
\begin{tikzcd}
A \arrow[r, hook] \arrow[dr, "s'"'] & X \arrow[d, dashed, "S'"] & X^{\alpha_i} \arrow[l, "f_i"'] \arrow[d, dashed, "{(id, S'_*)}"] \\
& Y^V & X^{\alpha_i} \times (Y^V)^{\alpha_i} \arrow[l, "h_i"]
\end{tikzcd}
$$

commutes. Define a map $S : V \times X \to Y$, where $S(v, x) = S'(x)(v)$. Now the diagram

$$
\begin{tikzcd}
V \times A \arrow[r, hook] \arrow[dr, "s"'] & V \times X \arrow[d, dashed, "S"] & V \times X^{\alpha_i} \arrow[l, "id \times f_i"'] \arrow[d, dashed, "{(id, S_\circ)}"] \\
& Y & V \times X^{\alpha_i} \times Y^{\alpha_i} \arrow[l, "g_i"]
\end{tikzcd}
$$

commutes. $\square$

Interestingly the uniqueness of the function S can be proven by backtracking the arguments with diagrams. No induction argument is needed. This makes the proof diagrammatic.

Example 1.10. Let S be the follower function of $\mathbb{N}$. Define functions $+' : \mathbb{N} \times \{0\} \to \mathbb{N}, (n, 0) \overset{+'}{\longmapsto} n$ and $g : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}, (k, m, n) \overset{g}{\mapsto} S(n)$. By Theorem 1.9 there exists a unique function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ where $n + 0 = n$ and $k + S(n) = S(k + n)$ for all $k, n \in \mathbb{N}$. This function is called the addition of natural numbers. Similarly one is able to define multiplication.

In some cases the collection generated by the closure properties can be constructed recursively on natural numbers.

Theorem 1.11 (Recursive Construction for Closure)**.** *Let X and I be collections and let $n_i \in \mathbb{N}$ for $i \in I$. Fix a relation $R_i \subset X^{n_i} \times X$ for $i \in I$. Then*

$$
cl(R_i)_{i \in I} = \cup_{k \in \mathbb{N}} F_k
$$

where

$$
\begin{cases}
F_0 = \cup_{i \in I} R_i[\emptyset^{n_i}] \\
F_{k+1} = \cup_{i \in I} R_i[F_k^{n_i}] \cup F_k, k \in \mathbb{N}.
\end{cases}
$$

Proof. The claim follows from two simple induction arguments. $\square$

Corollary 1.12. *Let X, I be collections and let $n_i \in \mathbb{N}$ for $i \in I$. Assume that $A \subset X$ and $f_i : X^{n_i} \rightharpoonup X$ is a partial function for $i \in I$. Denote with B the collection recursively defined by $(A, f_i)_{i \in I}$. Let collections F_n be as in Theorem 1.11. Let $b \in B$. Then there exists a finite sequence $(b_i)_{i \leq n}$ where $b_0 \in A$, $b_n = b$, $b_i \in F_i$ and $b_{i+1} = f_j(b')$ for some $j \in I$ and $b' \in F_i^{n_j}$ for all $i < n$. Furthermore, if B is injective-recursively defined the sequence $(b_i)_{i \leq n}$ is unique.*

Proof. The existence of the sequence follows from Theorem 1.11 and uniqueness, when B is injective-recursively defined, follows from a short induction. □

We call such a sequence $(b_i)_{i\leq n}$ associated to b a construction of b from A with closure operations $(f_i)_{i\in I}$.

Example 1.13. Let A be a subset of a group G. Then A generates the smallest subgroup of G that contains A by taking all the possible finite combinations of elements of the form g, g^{-1} where $g \in A \cup \{e_G\}$.

1.2 Predicate logic

We are going to very quickly formulate first order predicate logic. This is done by fixing symbols and creating injective-recursively a collection of words, formulae, from the symbols. In a suitable context, we are able to define a meaning to these formulae. First order predicate logic gives a very general way to study different mathematical structures and hence is of interest in category theory.

1.2.1 Alphabet and model

Denote with $\mathcal{V}$ a countable infinite collection, which we shall now call the collection of variables. Furthermore, with $\mathcal{S}$ we denote the collection $\{\forall v, =, (,), \rightarrow, \neg, , \mid v \in \mathcal{V}\}$. Notice that S contains the symbol comma as an element.

Definition 1.14 (Alphabet)**.** Let $\mathcal{C}, \mathcal{F}$ and $\mathcal{R}$ be collections and let $\mathrm{ord} : \mathcal{F} \sqcup \mathcal{R} \rightarrow \mathbb{N}_1$ be a function. Then the tuple $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ is called an alphabet. Furthermore, the disjoint union $\Sigma = \mathcal{V} \sqcup \mathcal{S} \sqcup \mathcal{C} \sqcup \mathcal{F} \sqcup \mathcal{R}$ is called the collection of symbols of the alphabet L. The elements of $\mathcal{C}, \mathcal{F}, \mathcal{R}$ are called constant -, function -and relation symbols, respectively. The value $\mathrm{ord}(g)$ is called the order of the symbol $g \in \mathcal{F} \sqcup \mathcal{R}$.

Definition 1.15 (Model)**.** Let M be a collection, $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ an alphabet and let

$$T : \mathcal{C} \sqcup \mathcal{F} \sqcup \mathcal{R} \rightarrow M \cup \bigcup_{n\in\mathbb{N}} (M^{M^n} \cup \mathcal{P}(M^n)) \text{ be a function.}$$

Assume that $T(c) \in M, T(f) : M^{\mathrm{ord}(f)} \rightarrow M$ and $T(R) \subset M^{\mathrm{ord}(R)}$ for all $c \in \mathcal{C}, f \in \mathcal{F}$ and $R \in \mathcal{R}$. Then we call the pair $\mathcal{M} := (M, T)$ a model for the alphabet L. We call the set M the universe of the L-model $\mathcal{M}$ and the function T is called an L-interpretation on the collection M. We denote $T(c), T(f), T(R)$ by $c^{\mathcal{M}}, f^{\mathcal{M}}$ and $R^{\mathcal{M}}$, respectively.

We choose a convention for an alphabet $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ where $\mathcal{C}, \mathcal{F}, \mathcal{R}$ are finite. We will denote L by

$$(c_0, \ldots, c_k, f_0^{n_0}, \ldots, f_l^{n_l}, R_0^{m_0}, \ldots, R_p^{m_p}), \tag{1.1}$$

where the symbols c, f and R refer to constant -, function -and relation symbols in the context of alphabets, respectively. The upper indices define the order of the symbol. Furthermore, a model (M, T) for L is then denoted as a tuple

$$(M, T(c_0), \ldots, T(c_k), T(f_0^{n_0}), \ldots, T(f_l^{n_l}), T(R_0^{m_0}), \ldots, T(R_p^{m_p})).$$

It's very easy to give examples of models of alphabets. For example, if $L = (c, f^2)$, then any monoid $(M, e, +)$ is a model for L. We would like to be able to specify those L-models that define a monoid, for example. For this reason we are going to give a formal definition of a formula in the first order predicate logic.

Definition 1.16. Let $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ be an alphabet and assume that $\mathcal{M}$ and $\mathcal{N}$ are L-models with universes M and N, respectively. We say that a function $S : M \rightarrow N$ is a model morphism $\mathcal{M} \rightarrow \mathcal{N}$, if

$$\begin{aligned} S(c^{\mathcal{M}}) &= c^{\mathcal{N}}, c \in \mathcal{C} \\ S \circ f^{\mathcal{M}} &= f^{\mathcal{N}} \circ S_*, f \in \mathcal{F} \\ S_*[R^{\mathcal{M}}] &\subset R^{\mathcal{N}}, R \in \mathcal{R} \end{aligned}$$

where $S_{*,n} : M^n \to N^n, n \in \mathbb{N}$ and $S_{*,n}(x_1, \ldots, x_n) = (S(x_1), \ldots, S(x_n))$ and we may leave the number n implicit. The function S is said to be an isomorphisms, if it has an inverse, which is also a model morphism. Furthermore, we call a model morphism $S : \mathcal{M} \to \mathcal{N}$ globally full, if

$$S_*[R^{\mathcal{M}}] = R^{\mathcal{N}} \cap \mathrm{im}(S_*).$$

for every relation symbol R in L.

If the alphabet L contains no relation symbols, then every L-model morphism is globally full. If $L = (c, f^2)$, then a function between monoids is a monoid homomorphism, if and only if it is an L-model morphism.

Theorem 1.17. *Let $f : \mathcal{M} \to \mathcal{N}$ be an L-model morphism. Then f is an isomorphism, if and only if f is globally full bijection.*

Proof. If f is an isomorphism, then the global fullness follows from the fact that f^{-1} is an L-model morphism. Assume then that f is a globally full bijection. Since f is injective, $f^{-1}(c^{\mathcal{N}}) = c^{\mathcal{M}}$ for all constant symbols c of L. Let α be function symbol of L. Now

$$f^{-1}\alpha^{\mathcal{N}} = \alpha^{\mathcal{M}} f_*^{-1}$$

follows directly from

$$f\alpha^{\mathcal{M}} = \alpha^{\mathcal{N}} f_*$$

and the fact that $(f_*)^{-1} = (f^{-1})_*$, which justifies the notation f_*^{-1}.

Let R be a relation symbol of L. Since $f_*[R^{\mathcal{M}}] = R^{\mathcal{N}}$ and f_* is a bijection, $f_*^{-1}[R^{\mathcal{N}}] = R^{\mathcal{M}}$. Therefore f^{-1} is a model morphism and hence f is an isomorphism. □

As a special case we attain a bijective monoid homomorphism is an isomorphism.

Definition 1.18 (Submodel)**.** Let $\mathcal{M} = (M, T)$ be an L-model and $A \subset M$. Assume that (A, T') is also an L-model. We say that (A, T') is a submodel of $\mathcal{M}$, if the inclusion $A \hookrightarrow M$ is an L-model morphism $(A, T') \to \mathcal{M}$.

Notice that in the definition of a submodel the L-interpretation function T' is uniquely defined on constant and function symbols, but not necessarily on relation symbols. By the Fundamental Theorem of Recursion 1.2 any intersection of submodels defines a submodel.

Definition 1.19. Let $\mathcal{M}$ be an L-model with universe M. Assume that $A \subset M$ is closed under the application of functions $f^{\mathcal{M}}$ and $c^{\mathcal{M}} \in A$ for constant and function symbols c and f in L. Then A has two possibly different L-model structures $\mathcal{A}$ and $\mathcal{B}$ that make it a submodel of $\mathcal{M}$, where $R^{\mathcal{A}} = \emptyset$ and $R^{\mathcal{B}} = R^{\mathcal{M}} \cap (A \times A)$ for relation symbols R in L. We call the models $\mathcal{A}$ and $\mathcal{B}$ the discrete and the full submodel of $\mathcal{M}$ defined by A, respectively. When we say that a subset defines a submodel structure, we mean that it defines the full submodel structure.

Corollary 1.20 (Generated submodel)**.** *Let $\mathcal{M}$ be an L-model. Then any subset A of the universe of $\mathcal{M}$ defines the smallest submodel $\mathcal{B}$ of $\mathcal{M}$ that contains A. Furthermore, every element of $\mathcal{B}$ can attained from $A \cup \{c^{\mathcal{M}} \mid c \text{ a constant symbol in } L\}$ by a finite application of functions $f^{\mathcal{M}}$.*

Proof. This follows directly from Corollary 1.12. □

We say that the subset $A \subset M$ generates the submodel $\mathcal{M}$ in the previous theorem.

Theorem 1.21 (Inverse-image submodel)**.** *Let $f : \mathcal{M} \to \mathcal{N}$ be an L-model morphism and let $\mathcal{B}$ be a submodel of $\mathcal{N}$, with universe B. Then $f^{-1}[B]$ defines a full submodel of $\mathcal{M}$, which we call the inverse image of the submodel $\mathcal{B}$.*

Proof. Let α be a function symbol and denote the order of α by $n \in \mathcal{N}$ and denote $A = f^{-1}[B]$. Clearly $c^{\mathcal{M}} \in A$ for all constant symbols c of L. It suffices to show that $\alpha^{\mathcal{M}}[A^n] \subset A$. This holds, since

$$\begin{aligned} f[\alpha^{\mathcal{M}}[A^n]] &= (f\alpha^{\mathcal{M}})[A^n] \\ &= (\alpha^{\mathcal{N}} f_*)[A^n] \\ &\subset \alpha^{\mathcal{N}}[B^n] \\ &\subset B \end{aligned}$$

and hence $\alpha^{\mathcal{M}}[A^n] \subset A$. □

Definition 1.22 (Image of a submodel)**.** Let $f : \mathcal{M} \to \mathcal{N}$ be an L-model morphism. Let us denote the universe of $\mathcal{N}$ by N. We define the image $\mathcal{S}$ of f as a submodel of $\mathcal{N}$, where $f \mid : \mathcal{M} \to \mathcal{S}$ is a globally full surjection.

The definition of the image of an L-model morphism f is well-formulated: The image of f is uniquely specified by the condition that f is a globally full surjection. Since f is a model morphism, the set theoretic image of f contains the elements $c^{\mathcal{N}}$ for constant symbols c in L. Additionally, the set theoretical image of f is closed under the application of functions $g^{\mathcal{N}}$ for function symbols g in L. Hence the image of f exists.

Theorem 1.23 (Equalizer submodel)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be L-models and let $f, g : \mathcal{M} \to \mathcal{N}$ be L-model morphisms. Denote the universe of $\mathcal{M}$ by M. Then the collection $A = \{x \in M \mid f(x) = g(x)\}$ defines a submodel of $\mathcal{M}$.*

Proof. Since $f = g$ on A, we have by the definition of f and g being model morphisms that $c^{\mathcal{M}} \in A$ for constant symbols c. For a function symbol α and $a \in A^{\mathrm{ord}(\alpha)}$, we have

$$f(\alpha^{\mathcal{M}}(a)) = \alpha^{\mathcal{M}}(f_*(a)) = \alpha^{\mathcal{M}}(g_*(a)) = g(\alpha^{\mathcal{M}}(a)).$$

Hence $\alpha^{\mathcal{M}}(a) \in A$. Thus A defines a submodel of $\mathcal{M}$. □

We call the model defined by f and g in the previous proof the equalizer submodel of f and g.

Definition 1.24. Let $\mathcal{M}_i$ be an L-model with universe M_i for $i \in I$. We define the model $\mathcal{M}$ with universe $M = \prod_{i\in I} M_i$ where

$$\begin{aligned} c^{\mathcal{M}} &= (c^{\mathcal{M}_i})_{i\in I}, \\ f^{\mathcal{M}}(x^1, \ldots, x^n) &= (f^{\mathcal{M}_i}(x_i^1, \ldots, x_i^n))_{i\in I}, \\ R^{\mathcal{M}} &= \{(x^1, \ldots, x^k) \in M^k \mid (x_i^1, \ldots, x_i^n) \in R^{\mathcal{M}_i}, i \in I\} \end{aligned}$$

for constant symbol c, function symbol f and relation symbol R with orders n and k, respectively.

The maps $pr_i : M \to M_i$, $x \mapsto x_i$ are called projections and they are model morphisms. Additionally, if the axiom of choice holds for the collections M_i and the collections M_i are non-empty, then the projections are globally full surjections.

1.2.2 Formulae of predicate logic

To be able to define the formulas of first order predicate logic we must first define terms.

Definition 1.25 (L-terms)**.** Let $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ be an alphabet. Let Σ the collection of all L-symbols. Denote $\Sigma^+ = \cup_{n\in\mathbb{N}}\Sigma^n$ which consists of all finite sequences of symbols in Σ, called words. We define the collection of L-terms to be the collection $T \subset \Sigma^+$ defined recursively by the closure operations

$$\begin{cases} \mathcal{V} \sqcup \mathcal{C} \subset T \text{ and} \\ f(a_1, \ldots, a_n) \in T \text{ for } f \in \mathcal{F}, n := \mathrm{ord}(f) \text{ and } a_1, \ldots, a_n \in T \end{cases}.$$

Definition 1.26 (L-formula)**.** Let $L = (\mathcal{C}, \mathcal{F}, \mathcal{R}, \mathrm{ord})$ be an alphabet. Denote the collection of L-symbols by Σ and the collection of L-terms by T. We define the collection $\Psi \subset \Sigma^+$ of L-formulas to be the collection defined recursively by the closure operations

$$\begin{aligned} t = s \in \Psi &\text{ for } t, s \in T, \\ R(t_1, \ldots, t_n) \in \Psi &\text{ for } R \in \mathcal{R}, \mathrm{ord}(R) = n \text{ and } t_1 \ldots t_n \in T \text{ and lastly} \\ \neg\phi, (\phi \to \psi), \forall v\phi \in \Psi &\text{ for } \phi, \psi \in \Psi \text{ and } v \in \mathcal{V}. \end{aligned}$$

We call the formulas $t = s, R(t_1, \ldots, t_{ord(R)})$ atomic formulas of Ψ for relation symbols $R \in \mathcal{R}$ and terms $t, s, t_i \in T, i \leq \mathrm{ord}(R)$.

From now on we fix an alphabet L, its symbols Σ, terms $\mathcal{T}$ and formulas Ψ. We denote by Σ^+ the free monoid over Σ.

Definition 1.27. Let $\psi_i \in \Psi, i \in \mathbb{N}, v \in \mathcal{V}$. We define the following conjunction $(\phi_1 \wedge \phi_2)$, disjunction $(\phi_1 \vee \phi_2)$ and existential quantification $\exists v\phi$ to denote formulas $\neg(\phi_1 \to \neg\phi_2), (\neg\phi_1 \to \phi_2)$ and $\neg\forall v\neg\phi$. We denote the associated functions by $Q_\wedge, Q_\vee, Q_{\forall v}$.

Theorem 1.28. *The collection of terms $\mathcal{T}$ and the collection of formulas Ψ are injective-recursively defined.*

Proof. This follows from Theorem 1.6. ☐

Truth in first order logic

Definition 1.29 (Algebraic, existential, universal and positive formulas)**.** In the following we denote by $\forall$ and $\exists$ as a shorthand for the collections of connectives $(\forall v)_{v\in\mathcal{V}}$ and $(\exists_v)_{v\in\mathcal{V}}$. The formulas created from the atomic formulas of alphabet L only using only connectives

1. $(\wedge, \forall)$ (algebraic)
2. $(\wedge, \vee, \exists)$ (existential)
3. $(\wedge, \vee, \forall)$ (universal)
4. $(\wedge, \vee, \exists, \forall)$ (positive)

are called algebraic, existential, universal and positive formulas, respectively. Additionally, if A is a collection of connectives, then we say that a formula ϕ is an A-formula, if ϕ is attained from the atomic formulas by finite application of connectives in A.

Definition 1.30. Let $\mathcal{M} = (M, T)$ be an L-model. We call a function $s : \mathcal{V} \to M$ an assignment of the model $\mathcal{M}$. By Theorem 1.7, s extends uniquely to a function $S : \mathcal{T} \to M$, where

$$S(v) = s(v)$$
$$S(f(t_1, \ldots, t_n)) = f^{\mathcal{M}}(S(t_i), \ldots, S(t_n))$$

for all $v \in \mathcal{V}, f \in \mathcal{F}$ and $t_i \in T, i \leq n$, where $n = \mathrm{ord}(f)$. We will use s to denote both functions S and s.

Notice that, if $f : \mathcal{M} \to \mathcal{N}$ is a model morphism and $s : \mathcal{V} \to M$ is a valuation on $\mathcal{M}$, then fs is a valuation on $\mathcal{N}$ and fs extends to $S' : \mathcal{T} \to N$. Denote the extension of s by S. Since f is a model morphism, then by an easy induction we see that $S' = fS$.

Definition 1.31 (Tarski's definition of truth)**.** Let $\mathcal{M} = (M, T)$ be a non-empty L-model and $\phi \in \Psi$. For an assignment $s : \mathcal{V} \to M$ we define $s_f : \mathcal{V} \to M$ for $f : \mathcal{V} \rightharpoondown M$, where

$$s_f(v) = \begin{cases} f(v), , \text{ if } v \in \mathrm{Dom}(f) \\ s(v), \text{ else.} \end{cases}$$

By $s_{(v,m)}, v \in \mathcal{V}, m \in M$ we mean s_f where f is the function $v \mapsto m : \{v\} \to M$. By Theorem 1.9 we may define a function $S_{\mathcal{M}} = S : M^{\mathcal{V}} \times \Psi \to \{0, 1\}$ by the following recursion:

$$S(s, t_1 = t_2) = \begin{cases} 1, \text{ if } s(t_1) = s(t_2) \\ 0, \text{ else} \end{cases}$$
$$S(s, R(t_1, \ldots, t_n)) = \begin{cases} 1, , \text{ if } (s(t_1), \ldots, s(t_n)) \in R^{\mathcal{M}} \\ 0, \text{ else} \end{cases}$$
$$S(s, \neg\phi) = 1 - S(s, \phi)$$
$$S(s, (\psi \to \phi)) = 1 - S(s, \psi) + S(s, \psi)S(s, \phi)$$
$$S(s, \forall v\phi) = \begin{cases} 1, , \text{ if } S(s_{(v,a)}, \phi) = 1 \text{ for all } a \in M \\ 0, \text{ else} \end{cases}$$

We say that the formula $\phi \in \Psi$ is true in model $\mathcal{M}$ with respect to an assignment s on $\mathcal{M}$, if $S_{\mathcal{M}}(s, \phi) = 1$ and denote $\mathcal{M} \vDash_s \phi$. If $\mathcal{M} \vDash_s \phi$ for all assignments s on $\mathcal{M}$, we denote $\mathcal{M} \vDash \phi$ and say that ϕ is true in model $\mathcal{M}$. If $T \subset \Phi$, then by $\mathcal{M} \vDash T$ we mean that $\mathcal{M} \vDash \phi$ for every $\phi \in T$.

Assume that L-model $\mathcal{N}$ has an empty universe. Let ϕ and ψ be formulas. We define $V : \Psi \to \{0,1\}$ by the following recursion: Let $\phi, \psi \in \Psi$ and $v \in \mathcal{V}$.

$$\begin{aligned} V(\phi) &= 1, \text{ if } \phi \text{ is an atomic formula} \\ V(\neg\phi) &= 1 - V(\phi) \\ V(\psi \to \phi) &= 1 - V(\psi) + V(\phi)V(\psi) \\ V(\forall v\phi) &= 1 \end{aligned}$$

We say that the empty L-model $\mathcal{N}$, if it exists, satisfies ϕ, if $V(\phi) = 1$. Then we denote $\mathcal{N} \vDash \phi$. We incorporate the same language for empty models as with non-empty models for the truth of a formula.

Example 1.32.

1. Let $L = \{c, f^2\}$. Let $x, y, z \in V$ be different variables and define T to be the collection consisting of the following three formulas

$$\begin{aligned} \forall x\forall y\forall z f(x, f(y,z)) &= f(f(x,y),z), \\ \forall x f(x,c) &= x \text{ and} \\ \forall x f(c,x) &= x. \end{aligned}$$

   Any L-model $\mathcal{M}$ is a monoid, if and only if $\mathcal{M} \vDash T$.

2. Now we give the axioms for a left R-module. First we fix the alphabet $L = (c, f^1, +^2, (f_r^1)_{r\in R}))$. Let x, y, z be different variables. Denote the neutral element of multiplication of R by 1_R. By the formula $(a+b)$ we mean $+(a,b)$. The unnecessary brackets are dropped. Now the following formulas for every $r, r' \in R$ are called the axioms of an $R-$module:

   - $\forall x\forall y\forall z(x+y)+z = x+(y+z)$
   - $\forall x(x+c) = x$
   - $\forall x\forall y(x+y) = (y+x)$
   - $\forall x f(x) + x = c$
   - $\forall x\forall y f_r(x+y) = f_r(x) + f_r(y)$
   - $\forall x f_{r+r'}(x) = f_r(x) + f_{r'}(x)$
   - $\forall x f_{1_R}(x) = x$
   - $\forall x f_r(f_{r'}(x)) = f_{rr'}(x)$

   Notice that the collection of an R-module axioms are indexed by the elements r, r' of R. Therefore there could be unaccountably many formulas. Denote the collection of R-module axioms by T. Now any model $\mathcal{M}$ of the alphabet L is a R-module, if and only $\mathcal{M} \vDash T$. Additionally, any function between two R-modules is R-linear, if and only if it is $L-$model morphism. The formulas in the left column are called the axioms of an abelian groups. Notice that all the R-module-axioms are algebraic. This turns out to be a very interesting property and we will prove very general statements in algebraic context.

3. Consider the alphabet $L = (R^2)$ and the following list of L-formulas, where x, y, z are different variables:

   - $\forall x R(x,x)$
   - $\forall x \neg R(x,x)$
   - $\forall x\forall y(R(x,y) \to R(y,x))$
   - $\forall x\forall y((R(x,y) \wedge R(y,x)) \to x = y)$
   - $\neg\exists x\exists y(R(x,y) \wedge R(y,x))$
   - $\forall x\forall y\forall z((R(x,y) \wedge R(y,z)) \to R(x,z))$
   - $\forall x\forall y(R(x,y) \vee R(y,x))$
   - $\forall x\forall y((R(x,y) \vee x = y) \vee R(y,x))$

   The formulas in the left column are called in order the axioms of reflexivity, anti-reflexivity, symmetry and anti-symmetry. The formulas on the right column are called the axioms of strict anti-symmetry, transitivity, linearity and strict linearity (trichotomy).

   Let $\mathcal{M} = (M, S)$ be an L-model.

   (a) The relation S is called a preorder, if $\mathcal{M}$ satisfies the axioms of reflexivity and transitivity. The model $\mathcal{M}$ is then called a pre-ordered set and a proset.

   (b) We call the relation S an equivalence relation, if $\mathcal{M}$ is a proset that satisfies symmetry.

(c) If $\mathcal{M}$ is a proset that satisfies anti-symmetry, we call $\mathcal{M}$ a partially ordered set and a poset. Additionally, the relation S is called a partial order.

(d) If $\mathcal{M}$ is a poset that satisfies linearity, we call $\mathcal{M}$ a linearly ordered set and the relation S is called a linear order.

In the context of orders, we denote the relation S by $\leq$. Consider a relation $\leq$ on the collection M. We define $<:=\leq \setminus \{(x,x) \mid x \in M\}$, $\geq:=\leq^{-1}$ and $>:=<^{-1}$.

It is worth noting, that many of the formulas relating to order are not even positive formulas.

Definition 1.33. Let $\mathcal{M} = (M, \leq)$ be a poset. Let $A \subset M$ and $x \in M$.

1. The element x is called an upper bound for A, if $a \leq x$ for every $a \in A$.
2. The element x is called the maximum of A, if x is an upper bound for A and $x \in A$.
3. The element x is called a maximal element of A, if there doesn't exists $y \in A$ where $x < y$.

Notice that $(M, \geq)$ is a poset. We say that x is a lower bound, minimum element and minimal element of A, if x is an upper bound, maximum element and maximal element of A in $(M, \geq)$, respectively. Lastly, we say that x is the supremum of A, if it is the minimum element of the upper bounds of A. Dually we say that x is the infimum of A, if x is the supremum of A in $(M, \geq)$.

Notice that the definition of a supremum isn't easily turned into first order formula, because we need to quantify over subcollections of our universe.

Lemma 1.34. *Let L be an alphabet. Then the following holds:*

1. *If L contains no constant symbols, then there exists a unique L-model $\emptyset$, whose universe is the empty set. Furthermore, $\emptyset$ satisfies all $(\wedge, \vee, \rightarrow, \forall)$-formulas and the unique map from $\emptyset$ to a L-model $\mathcal{N}$ is a model morphism.*
2. *Define the singleton model 1 of L where the universe consists of a single element and whose interpretations of relation symbols are the maximal relations. Then 1 satisfies all $(\wedge, \vee, \rightarrow, \exists, \forall)$-formulae. Furthermore, the unique function from the L-model $\mathcal{N}$ to 1 is a model morphism.*

Proof.

1. The uniqueness and existence of the empty L-model is clear, since no constants exists in L. Directly from the definition of truth of a formula, we see that the $(\wedge, \vee, \rightarrow, \exists, \forall)$-formulas are true in the empty model. Vacuously the unique map from the empty model to an other L-model is a model morphism.
2. Fixing a singleton set 1, we notice that the interpretations of constant symbols and function symbols are uniquely defined. We define the interpretation of a R^n to be the whole set 1^n for all relation symbols R^n in L. This defines the singleton L-model 1.

   We see that $1 \vDash \phi$ for all atomic ϕ. Furthermore, if 1 satisfies ϕ and ψ, then clearly 1 satisfies $\phi \wedge \psi$, $\phi \vee \psi$,$\phi \rightarrow \psi$, $\exists v \phi$ and $\forall \phi$.

   Lastly, the unique map from the L-model $\mathcal{N} \rightarrow 1$ is clearly a model morphism.

$\square$

Definition 1.35 (Free variables)**.** Let $v \in \mathcal{V}$. Now we define a function $F : \mathcal{T} \sqcup \Psi \rightarrow \mathcal{P}(\mathcal{V})$ by

$$\begin{aligned}
\mathrm{Free}(v) &= \{v\} \\
\mathrm{Free}(f(t_1, \ldots, t_n)) &= \cup_{i \leq n} \mathrm{Free}(t_i) \\
\mathrm{Free}(t_1 = t_2) &= \mathrm{Free}(t_1) \cup \mathrm{Free}(t_2) \\
\mathrm{Free}(R(t_1, \ldots, t_n)) &= \cup_{i \leq n} \mathrm{Free}(t_i) \\
\mathrm{Free}(\neg \phi) &= \mathrm{Free}(\phi) \\
\mathrm{Free}((\psi \rightarrow \phi)) &= \mathrm{Free}(\psi) \cup \mathrm{Free}(\phi) \\
\mathrm{Free}(\forall v \phi) &= \mathrm{Free}(\phi) \setminus \{v\}
\end{aligned}$$

for $n \in \mathbb{N}, v \in \mathcal{V}, t_i \in T$, for $i \in \mathbb{N}, f \in \mathcal{F}, R \in \mathcal{R}$ and $\phi, \psi \in \Psi$ where $\mathrm{ord}(f) = \mathrm{ord}(R) = n$. Let $\phi \in \Phi$. We say that $\mathrm{Free}(\phi)$ is the collection of free variables of ϕ.

Definition 1.36 (Sentence and theory)**.** Let T be a collection of L-formulas and let ϕ be an L-formula. We say that ϕ is a sentence, if ϕ has no free variables. Moreover, we say that T is an L-theory, if every formula in T is a sentence. We call a theory T algebraic, universal, positive and existential, if all the formulas in T are algebraic, universal, positive and existential, respectively.

Lemma 1.37. *Let $\mathcal{M} = (M, T)$ be a non-empty L-model, $\phi \in \Psi$ and let s and s' be valuations on $\mathcal{M}$. Let A be a collection of variables and assume that s and s' agree on A. If Free$(\phi) \subset A$, then $S_{\mathcal{M}}(s, \phi) = S_{\mathcal{M}}(s', \phi)$.*

Proof. First we prove that $s(t) = s'(t)$ for all terms t, where $\text{Free}(t) \subset A$. Let

$$I := \{t \in \mathcal{T} \mid \text{ If Free}(t) \subset A, \text{ then } s(t) = s'(t).\}.$$

Clearly $\mathcal{V} \subset I$. Let $t_1, \dots t_n \in I$ and $f \in \mathcal{F}, \text{ord}(f) = n$. Assume that $\text{Free}(f(t_1), \dots, f(t_n)) = \cup_{i \leq n}\text{Free}(t_i) \subset A$. Now

$$s(f(t_1, \dots, t_n)) = T(f)(s(t_1), \dots, s(t_n))) = T(f)(s'(t_1), \dots, s'(t_n)) = s'(f(t_1, \dots, t_n)).$$

Hence the claim holds for terms.

Denote then with J the collection of formulas that satisfy the claim. Let $R \in \mathcal{R}, n := \text{ord}(R)$ and $t_i \in \mathcal{T}$ for $i \in \mathbb{N}$. Assume that s and s' agree on the free variables of $t_1 = t_2$. Now since $s(t_1) = s'(t_1)$ and $s(t_2) = s'(t_2)$, we attain $t_1 = t_2 \in J$. Similarly $R(t_1, \dots, t_n) \in J$.

Let $\phi, \psi \in J$ and assume that valuations s and s' agree on free variables of the formula $\neg\phi$. Since $\phi \in J$,

$$S_{\mathcal{M}}(s, \neg\phi) = 1 - S_{\mathcal{M}}(s, \phi) = 1 - S_{\mathcal{M}}(s', \phi) = S_{\mathcal{M}}(s', \neg\phi)$$

Therefore $\neg\phi \in J$. After a similar computation we see that $(\phi \to \psi) \in J$. Assume lastly that valuations s and s' agree on the free variables of $\forall v\phi$. Now the valuations $s_{(v,m)}$ and $s'_{(v,m)}$ agree on the free variables of ϕ for $m \in M$. By assumption then $S(s_{(v,m)}, \phi) = S(s'_{(v,m)}, \phi)$ for all $m \in M$. Hence $S_{\mathcal{M}}(s, \forall v\phi) = S_{\mathcal{M}}(s', \forall\phi)$ and $\forall v\phi \in J$. $\square$

Corollary 1.38. *Let $\mathcal{M}$ be a non-empty L-model and ϕ an L-sentence. Then $\mathcal{M} \vDash \phi$, if and only there exists a valuation s on $\mathcal{M}$ where $\mathcal{M} \vDash_s \phi$.*

Proof. This follows from Lemma 1.37. $\square$

Fix a formula ϕ and denote by A the collection of free variables of ϕ. Assume that $v \in A$. Notice that for any fixed L-model $\mathcal{M} = (M, T)$, ϕ defines a function

$$M^A \to \{0, 1\}, f \mapsto S_{\mathcal{M}}(s_f, \phi) =: \phi^{\mathcal{M}}(f) =: \phi(f), \text{ for any valuation } s \text{ on } \mathcal{M}.$$

This function is well-defined by Lemma 1.37. For any $f : V \setminus \{v\} \to M$, we define $\phi(v, f)$ to denote the function $M \to \{0, 1\}$, where $\phi(v, f)(x) =: \phi(x, f) = S_{\mathcal{M}}(s_{(v,x),f}, \phi)$.

Definition 1.39 (Definable subcollections and class)**.** Let $\mathcal{M} = (M, T)$ be an L-model and let ϕ be a formula with A as the collection of free variables. Assume that $v \in A$. Let $f : A \setminus \{v\} \to M$. We call the collection C over which $\phi(v, f)$ becomes a characteristic function a definable subcollection of the universe of $\mathcal{M}$, with formula ϕ and variables f. Additionally, we use the term class for the subcollection C.

Preservation of truth

From now on the models are assumed to be non-empty.

Definition 1.40. Let $f : \mathcal{M} \to \mathcal{N}$ be an L-model morphism, $\phi \in \Phi$ and s an assignment on $\mathcal{M}$. We say that f preserves the truth of ϕ with respect to the assignment s, if $\mathcal{M} \vDash_s \phi$ implies that $\mathcal{N} \vDash_{f \circ s} \phi$. Additionally, we say that f reflects the truth of ϕ with respect to the assignment s, if $\mathcal{N} \vDash_{f \circ s} \phi$ implies that $\mathcal{M} \vDash_s \phi$. We say that f preserves the truth of ϕ, if f preserves the truth of ϕ for every assignment s on $\mathcal{M}$ and similarly for reflection of the truth of ϕ.

Notice that, if $f : \mathcal{M} \to \mathcal{N}$ is a model morphism that preserves the truth of a sentence ϕ for some valuation, then f preserves the truth of ϕ by Lemma 1.37.

Theorem 1.41. *Let $\mathcal{M} = (M, T)$ and $\mathcal{N} = (N, V)$ be L-models and let $f : \mathcal{M} \to \mathcal{N}$ be a model morphism. Then the following claims hold:*

1. *The model morphism f preserves the truth of existential formulas.*
2. *If f is surjective, then f preserves the truth of positive formulas.*
3. *If f is a globally full injection, then f reflects the truth of all universal formulas.*
4. *If f is an isomorphism, then f preserves the truth of all formulas.*

Proof. Let

$$I = \{\phi \in \Phi \mid S_{\mathcal{M}}(s, \phi) \leq S_{\mathcal{N}}(f \circ s, \phi) \text{ for all valuations } s \text{ on } \mathcal{M}\} \text{ and}$$
$$J = \{\phi \in \Phi \mid S_{\mathcal{M}}(s, \phi) \geq S_{\mathcal{N}}(f \circ s, \phi) \text{ for all valuations } s \text{ on } \mathcal{M}\}$$

.

1. We prove the claim by induction on the structure of existential formulas. Let $\phi, \psi \in I, v \in \mathcal{V}, R \in \mathcal{R}$ and t_i a term for $i \in \mathbb{N}$. Fix a valuation s on $\mathcal{M}$ and denote $n := \operatorname{ord}(R)$. We need to show that $t_1 = t_2, R(t_1, \dots, t_n) \in I$ and $(\phi \wedge \psi), (\phi \vee \psi), \exists v\phi \in I$.

   We may assume that $S_{\mathcal{M}}(s, t_1 = t_2) = 1$. Hence $s(t_1) = s(t_2)$. Thus $fs(t_1) = fs(t_2)$ and thus $S_{\mathcal{N}}(fs, t_1 = t_2)$. Thus $t_1 = t_2 \in I$. Again we may assume that $S_{\mathcal{M}}(s, R(t_1, \dots, t_n)) = 1$. Thus $(s(t_1), \dots, s(t_n)) \in R^{\mathcal{M}}$ and, since f is a model morphism, $(fs(t_1), \dots, fs(t_n)) \in R^{\mathcal{N}}$. Hence $S_{\mathcal{N}}(fs, R(t_1, \dots, t_n)) = 1$ and $R(t_1, \dots, t_n) \in I$.

   Assume that $S_{\mathcal{M}}(s, (\phi \wedge \psi)) = 1$. Thus $S_{\mathcal{M}}(s, \phi) = 1$ and $S_{\mathcal{M}}(s, \phi) = 1$. Because $\phi, \psi \in I$, we attain $S_{\mathcal{N}}(fs, \phi) = S_{\mathcal{N}}(fs, \psi) = 1$. Hence $S_{\mathcal{N}}(fs, (\phi \wedge \psi)) = 1$ and so $(\phi \wedge \psi) \in I$. Similarly $(\phi \vee \psi) \in I$. Now assume that $S_{\mathcal{M}}(s, \exists v\phi) = 1$. Thus there exists $m \in M$ where $S_{\mathcal{M}}(s_{(v,m)}, \phi) = 1$. Since $\phi \in I$, $S_{\mathcal{N}}(f \circ (s_{(v,m)}), \phi) = 1$ and so $S_{\mathcal{N}}((f \circ s)_{(v,f(m))}, \phi) = 1$, we have $S_{\mathcal{N}}(fs, \exists v\phi) = 1$ and thus $\exists v\phi \in I$. Thus every existential formula is contained in I.

2. Assume then that f is surjective and $\phi \in I$ and $v \in \mathcal{V}$. It suffices to show that $\forall v\phi \in I$. Let s be a valuation on $\mathcal{M}$. Assume that $S_{\mathcal{M}}(s, \forall v\phi) = 1$. Thus $S_{\mathcal{M}}(s_{(v,m)}, \phi) = 1$ for all $m \in M$. Since $\phi \in I$, we attain $S_{\mathcal{N}}((fs)_{(v,f(m))}, \phi) = 1$ for all $m \in \mathcal{M}$. By the surjectivity of f we attain $S_{\mathcal{N}}((fs)_{(v,n)}) = 1$ for all $n \in N$ and hence $S_{\mathcal{N}}(fs, \forall v\phi) = 1$. Thus $\phi \in I$.

3. Assume that f is a globally full injection. Let $\phi, \psi \in J, t_i \in \mathcal{T}$, for $i \in \mathbb{N}, R \in \mathcal{R}$ and $v \in \mathcal{V}$. Denote $n = \operatorname{ord}(R)$. We need to show that

   $$t_1 = t_2, R(t_1, \dots, t_n), (\phi \wedge \psi), (\phi \vee \psi), \forall v\phi \in J.$$

   We may assume that $S_{\mathcal{N}}(fs, t_1 = t_2) = 1$. Thus $fs(t_1) = fs(t_2)$. By the injectivity of f we have $s(t_1) = s(t_2)$ and so $S_{\mathcal{M}}(s, t_1 = t_2)$, whence $t_1 = t_2 \in J$. Assume next that $S_{\mathcal{N}}(fs, R(t_1, \dots, t_n)) = 1$. So $(fs(t_1), \dots, fs(t_n)) \in R^{\mathcal{N}}$. Since f is injective and $f_*[R^{\mathcal{M}}] = R^{\mathcal{N}} \cap \operatorname{im}(f_*)$, we have that $(s(t_1), \dots, s(t_n)) \in R^{\mathcal{M}}$. Hence $S_{\mathcal{M}}(s, R(t_1, \dots, t_n)) = 1$ and so $R(t_1, \dots, t_n) \in J$.

   Similarly as in case 1. we obtain $(\phi \wedge \psi)$, and $(\phi \vee \psi) \in J$. Assume that $S_{\mathcal{N}}(fs, \forall v\phi) = 1$. Thus $S_{\mathcal{N}}(fs_{(v,x)}, \phi)) = 1$ for all $x \in N$. Therefore

   $$S_{\mathcal{M}}(s_{(v,m)}, \phi) \geq S_{\mathcal{N}}(f \circ (s_{(v,m)}), \phi) = S_{\mathcal{N}}((fs)_{(v,f(m)}, \phi) = 1$$

   for all $m \in M$. Hence $S_{\mathcal{M}}(s, \forall v\phi) = 1$ and so $\forall v\phi \in J$.

4. Assume that f is an isomorphism. Now $f_*[R^{\mathcal{M}}] = R^{\mathcal{N}}$ for all relation symbols $R \in \mathcal{R}$. Let $\phi, \psi \in I \cap J$. It suffices to show that $\neg\phi, (\phi \to \psi) \in I \cap J$. Notice that $S_{\mathcal{M}}(s, \neg\phi) = 1 - S_{\mathcal{M}}(s, \phi) = 1 - S_{\mathcal{N}}(s, \phi) = S_{\mathcal{N}}(s, \neg\phi)$ for all valuations s on $\mathcal{M}$. So $\neg\phi \in I \cap J$. Similarly $(\phi \to \psi) \in I \cap J$.

□

Theorem 1.42. *Let $\mathcal{M}_i$ be an L-model with a universe M_i for $i \in I$. Assume the choice assumption: For any non-empty collections $A_i \subset M_i, i \in I$, the product $\prod_{i \in I} A_i$ is non-empty. Denote with $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$. Then for any $(\wedge, \exists, \forall)$-formula ϕ holds that $\mathcal{M} \vDash \phi$, if $\mathcal{M}_i \vDash \phi$ for every $i \in I$.*

Proof. Denote the projections by $p_i : \mathcal{M} \to \mathcal{M}_i$. Let J be the collection of $(\wedge, \exists, \forall)$-formulas ϕ where, if $\mathcal{M}_i \vDash_{p_i \circ s} \phi$, then $\mathcal{M} \vDash_s \phi$ for every valuation s on $\mathcal{M}$. Fix a valuation s on $\mathcal{M}$. Let t_i, f and R be a term, a function symbol and a relation symbol for $i \in \mathbb{N}$, respectively. Additionally, fix $\phi \in J$ and a variable $v \in \mathcal{V}$.

If $S_{\mathcal{M}_i}(p_i s, t_1 = t_2) = 1$ for all $i \in I$, then $p_i s(t_1) = p_i s(t_2)$ for all $i \in I$ and thus $s(t_1) = s(t_2)$. Hence $S_{\mathcal{M}}(s, t_1 = t_2)$ and so $t_1 = t_2 \in J$. Similarly $R(t_1, \ldots, t_{\mathrm{ord}(R)}) \in J$.

Assume that $S_{\mathcal{M}_i}(p_i s, \exists v \phi) = 1$. Therefore for every $i \in I$ there exists $m_i \in M_i$ where $S_{\mathcal{M}_i}((p_i s)_{(v,m_i)}, \phi)$. By the choice assumption there exists $m \in M$ such that $S_{\mathcal{M}_i}(p_i \circ (s_{v,m}), \phi) = 1$ for every $i \in I$. Thus $S_{\mathcal{M}}(s(v,m), \phi) = 1$ and so $S_{\mathcal{M}}(s, \exists v \phi) = 1$. Therefore $\exists v \phi \in J$. Similarly $\forall v \phi \in J$, but the choice assumption is not needed. ☐

Definition 1.43 (L-congruence)**.** Let $\mathcal{M} = (M, T)$ be an L-model. An equivalence relation $\sim$ on M is called an L-congruence on the model $\mathcal{M}$, if the following property holds: Let $f \in \mathcal{F}$. Assume that $m_i \sim p_i$ for $i \leq \mathrm{ord}(f) =: n$. Then $f^{\mathcal{M}}(m_1, \ldots, m_n) \sim f^{\mathcal{M}}(p_1, \ldots, p_n)$.

Every function $f : X \to Y$ defines an equivalence relation $\ker(f) :=\sim_f$ by identifying those points that identically under f. The equivalence relation $\sim_f$ is called the set theoretic kernel of f or just the kernel of f. Let $\mathcal{M}$ be an L-model with a universe M. Any relation R on M generates the smallest L-congruence on $\mathcal{M}$ that contains R by The Fundamental Theorem of Recursion 1.2.

Lemma 1.44. *Let $\mathcal{M} = (M, T), \mathcal{N} = (N, V)$ and $\mathcal{P} = (P, W)$ be L-models and let $p : \mathcal{M} \to \mathcal{N}$ and $g : \mathcal{M} \to \mathcal{P}$ be model morphisms.*

1. *Assume p is a surjection. Then ker(p) $\subset$ ker(g), if and only if there exists a unique $\tilde{g} : N \to P$ that makes the diagram*

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "p"'] & P \\ N \arrow[ur, dotted, "\exists! \tilde{g}"'] & \end{tikzcd}$$

   commute. Additionally, $\tilde{g}$ is injective, if and only if ker(p) = ker(g). Lastly, if p is globally full, then $\tilde{g}$ is a model morphism.

2. *The kernel of p is an L-congruence. Conversely all L-congruences define canonically an L-model structure on $M/\sim$, where the L-structure on $M/\sim$ is uniquely defined by the requirement that the quotient map $q : M \to M/\sim$ is a globally full L-model morphism. Especially all L-congruences on $\mathcal{M}$ rise as a kernel of some globally full surjective L-model morphism h with $\mathcal{M}$ as its domain.*

3. *We have a unique factorization $g = \tilde{g} \circ q$ where q is the quotient map $M \to M/ker(f)$. Additionally, $\tilde{g}$ is an injective model morphism.*

Proof.

1. Assume that $\ker(p) \subset \ker(g)$. By the surjectivity of p the uniqueness is clear. Define $\tilde{g} : N \to P$ via $p(x) \mapsto g(x)$. This is well-defined by the assumption $\ker(p) \subset \ker(g)$. Clearly $\tilde{g}p = g$.

   Assume then that $\tilde{g}$ exists. If $p(x) = p(y)$, then

$$g(x) = \tilde{g}(p(x)) = \tilde{g}(p(y)) = g(y)$$

   and so the converse holds.

   Next we will show that $\tilde{g}$ is injective, if and only if $\ker(p) = \ker(g)$. Fix $x, x' \in M$. First assume that $\tilde{g}$ is injective. We show that $\ker(g) \subset \ker(p)$. If $g(x) = g(x')$, we get that $\tilde{g}(p(x)) = \tilde{g}(p(x'))$. By the injectivity of $\tilde{g}$, $p(x) = p(x')$ and so $\ker(p) = \ker(g)$. Assume then the converse $\ker(g) = \ker(p)$. Now, if $\tilde{g}(p(x)) = \tilde{g}(p(x'))$, then $g(x) = g(x')$ and so $p(x) = p(x')$. By the surjectivity of p we have the injectivity of $\tilde{g}$.

   Assume that $\tilde{g}$ exists and $p[R^{\mathcal{M}}] = R^{\mathcal{N}}$ for all relation symbols $R \in \mathcal{R}$. We claim that $\tilde{g}$ is a model morphism. Let $c \in \mathcal{C}, f \in \mathcal{F}$ and $R \in \mathcal{R}$. Now

$$\tilde{g}(c^{\mathcal{N}}) = \tilde{g}(p(c^{\mathcal{M}})) = g(c^{\mathcal{M}}) = c^{\mathcal{P}}.$$

Since $p_*[R^{\mathcal{M}}] = R^{\mathcal{N}}$, we have

$$\tilde{g}_*[R^{\mathcal{N}}] = \tilde{g}_*[p_*[R^{\mathcal{M}}]] = g_*[R^{\mathcal{M}}] \subset P(R).$$

Lastly,

$$\tilde{g}f^{\mathcal{N}}p_* = \tilde{g}pf^{\mathcal{M}} = gf^{\mathcal{M}} = f^{\mathcal{P}}g_* = f^{\mathcal{P}}(\tilde{g}p)_* = f^{\mathcal{P}}\tilde{g}_*p_*$$

By the surjectivity of p_* we obtain $\tilde{g}f^{\mathcal{N}} = f^{\mathcal{P}}\tilde{g}_*$. Thus $\tilde{g}$ is a model morphism.

2. Denote the kernel of p by $\sim$. Since $pf^{\mathcal{M}} = f^{\mathcal{N}}p_*$ for all $f \in \mathcal{F}$, we attain $\sim$ is an L-congruence on $\mathcal{M}$. Assume then that $\sim$ is a congruence on $\mathcal{M}$. Consider the quotient map $q : X \to X/\sim$. The kernel of q is $\sim$. We need to show that an L-structure can be given on the collection $M/\sim$. Define the interpretation function $\tilde{T}$ as follows: Let $c \in \mathcal{C}, f \in \mathcal{F}, R \in \mathcal{R}$. Denote $n = \mathrm{ord}(f)$ and define

$$\begin{aligned}\tilde{T}(c) &= q(T(c))\\ \tilde{T}(f) &= g'\\ \tilde{T}(R) &= q_*[R^{\mathcal{M}}]\end{aligned}$$

where g' is the unique function $g' : (M/\sim)^n \to M/\sim$ with $qT(f) = g'q_*$. Notice that q_* is a surjection and $\ker(q_*) \subset \ker(qT(f))$ and by part 1 $\tilde{T}$ is well-defined. By construction we see the uniqueness of $\tilde{T}$.

3. This is clear, since the quotient map has the same set theoretical kernel as the given function g. Hence we may apply the previous result.

□

Definition 1.45 (Quotient model)**.** Let $\mathcal{M}$ be an L-model and let $\sim$ be a congruence on $\mathcal{M}$. There exists a unique L-interpretation function $\tilde{T}$ on $M/\sim$ where the quotient $q : \mathcal{M} \to (M/\sim, \tilde{T})$ is a globally full model morphism. We call the model $(M/\sim, \tilde{T})$ a quotient model of $\mathcal{M}$ and denote it by $\mathcal{M}/\sim$.

Theorem 1.46 (Fundamental Theorem of Model Morphism)**.** *Let $\mathcal{M}$ and $\mathcal{N}$ be L-models and let $f : \mathcal{M} \to \mathcal{N}$ be a model morphism. Then there exists a unique model morphism $\tilde{f} : \mathcal{M}/\ker(f) \to \mathcal{N}$ where the diagram*

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, "q"] & \mathcal{N} \\ \mathcal{M}/ker(f) \arrow[ur, dotted, "\exists! \tilde{f}"'] & \end{tikzcd}$$

commutes. Additionally, $\tilde{f}$ is injective. Furthermore, $\mathcal{M}/ker(f) \vDash \phi$ for any positive sentence ϕ, where $\mathcal{M} \vDash \phi$. Lastly, if f is globally full, then $\mathcal{M}/ker(f) \vDash \phi$ for any universal sentence ϕ, where $\mathcal{N} \vDash \phi$.

Proof. This follows follows from Lemma 1.44 and Theorem 1.41). □

Notice that The Fundamental Theorem of Homomorphism in group theory follows from The Fundamental Theorem of Model Morphism, since the group axioms are positive sentences. Thus the quotient model morphism preserves the sentences. Similar theorems exist for R-modules, monoids, rings and monoid actions and they all follow from The Fundamental Theorem of Model Morphism.

1.3 Set theory

Set theory **ZFC** is an L-theory, where $L = \{R^2\}$ and R^2 refers to a relation symbol of order two. The axioms of **ZFC** are meant to characterize properties of the membership relation $\in$, but as it turns out a full characterization is impossible using first order predicate logic [7]. We assume that there exists an L-model $(V, \in)$ that satisfies the axioms of set theory, where the binary relation $\in$ is the membership relation. The collection V is said to be the collection of all sets and the elements of V are called sets. Since the formula $v = v$ defines V, V is a class. Let $C \subset V$ be a class. A function $f : C \to V$ is a called a class function, if the graph $\{(x, f(x)) \mid x \in C\}$ of f is a class of $(V, \in)$. Let X be a set. Consider the L-formula $\phi = R(y, x)$. Now the formula $\phi(y, X)$ defines a class which is exactly the set X. Thus all sets are classes.

We formulate the axioms of **ZFC** informally. It is straightforward to see that they can be expressed as $\{R^2\}$-sentences:

1. Axiom of extensionality: Two sets are the same, if and only if they share the exact same elements.
2. Axiom of unordered pair: For any two sets x and y there exists a set z that contains exactly x and y as the elements. The set z is denoted by $\{x, y\}$.
3. Axiom of subset: Given a set x and a class A, the intersection $x \cap A$ is a set denoted by $\{a \in x \mid a \in A\}$.
4. Axiom of union: Let x be a set. Then there exists a set, denoted $\cup x$, that is the union of the sets in x.
5. Axiom of power set: For any set x, there exists a set $\mathcal{P}(x)$, that contains exactly all subsets of x.
6. Axiom of infinity: There exists a set that has an injective but non-surjective map to itself.
7. Axiom of replacement: Let $F : V \to V$ be a class function. Then the image $F[A]$ of a set A is a set.
8. Axiom of regularity: Any non-empty set X has an element that is disjoint from X.
9. Axiom of choice: Let $\mathcal{A}$ be a set that doesn't have the empty set as an element. Then there exists a function $f : \mathcal{A} \to \cup\mathcal{A}$ where $f(A) \in A$ for all $A \in \mathcal{A}$.

We are able to code functions and Cartesian products, for example, as sets, as we did in definition 1.1. Set theory creates a unified way to view mathematics, but for our purposes of category theory we need to add one axiom. In category theory we would like to have a world that contains all sets and use tools such as the axiom of choice, but this will create foundational troubles. For this reason we limit ourselves to inside to a simulated universe of sets. Additionally, we want our theorems still to apply to all sets. Hence we will add the axiom of universes that implies the existence of a hierarchy of sets where each level gives us a model for set theory and every set belongs to some part of the hierarchy.

1.4 Well-order

To understand sets it is fundamental to understand order relations on sets. One reason is that we need to measure and compare different sizes of sets. More structured notion of a size is a specific kind of order called a well-ordering.

Definition 1.47 (Well-ordered set)**.** Let X be a set and let $\leq$ be a relation on X. We say that $\leq$ is a well-ordering of X, if it is a linear order and every non-empty subset of X has a minimum. The pair $(X, \leq)$ is then said to be a well-ordered set. If X is non-empty, we denote the minimum of X by 0. For every element $x \leq y$ we define the sets

- $[x, y] = \{a \in X \mid x \leq a \leq y\}$
- $(x, y] = \{a \in X \mid x < a \leq y\}$
- $[x, y) = \{a \in X \mid x \leq a < y\}$
- $(x, y) = \{a \in X \mid x < a < y\}$.

A subset A of X is called an initial segment of X, if $A = X$ or there exists $x \in X$, where $A = [0, x)$.

Let $x \in X$. We define x^+ to be the minimum of $X \setminus [0, x]$ whenever it exists and call it the follower element of x. If the element $x \in X$ follows no element, then x is called a limit element. If x is not a limit element, then it follows some unique element which we denote x^-.

The axiom of choice is a tool to study infinite sets and their interrelations. Famously the axiom of choice is equivalent with many seemingly different formulations. It's straightforward to see that the axiom of choice equivalent with the following statements:

1. Every surjective function has a right inverse function.
2. Let R be a relation from X to Y where for every $x \in X$ there exists $y \in Y$, where $(x, y) \in R$. Then there exists a function $f : X \to Y$ where $(x, f(x)) \in R$ for every $x \in X$.

We will prove that the following statements are equivalent with the axiom of choice:

1. Zorn's lemma: Let H be a non-empty poset. If every chain of H is bounded from above, then H contains a maximal element.
2. Well-ordering principle: Any set admits a well-ordering structure.

Notice that the well-ordering principle implies directly the axiom of choice: Let $\mathcal{A}$ be a set, where $\emptyset \notin \mathcal{A}$. By assumption, we may give a well-ordered structure to $\cup\mathcal{A}$. Then the choice function can be defined by taking the minimums of sets in $\mathcal{A}$.

Zorn's lemma also implies the axiom of choice: For a given surjection $f : X \to Y$, we may define a poset structure on those subset A of X, where $f \mid A$ is an injection, by inclusion of sets. By Zorn's lemma this poset has a maximal element A. Now $f \mid A$ is a bijection. Thus f has a right inverse, assuming the claim of Zorn's lemma.

1.5 Ordinals and cardinals

The cardinality of a set measures the place of the set in the hierarchy of sizes. Ordinals describe the position of an element in a well-ordered set. We would like to define cardinals as the equivalence classes that bijections define among the class of sets. This raises problems, since these classes are not sets in the formal set theory we choose to use. It so happens that we have a natural choice for the representatives of the classes of well-orders called the von Neumann ordinals that are constructed transfinitely from the empty set. Every cardinal can be identified with a certain ordinal.

Theorem 1.48 (Transfinite induction). *Let X be a well-ordered set. Let $I \subset X$ have the following property: For any $x \in X$, $[0, x) \subset I$ implies that $x \in I$. Then $I = X$.*

Proof. If I is not the whole set X, then there is the first element a not included in I. Thus the initial segment up to element a is in I and hence $a \in I$ which is a contradiction. Hence $I = X$. □

Definition 1.49. Let X be a set. The set X is called transitive, if $\cup X \subset X$. We say that X is an ordinal, if it is transitive and linearly ordered by $\in$-relation. The well-ordering relation is such that for $x, y \in X$, $x \leq y$, if and only if $x \in y$ or $x = y$. We denote the class of all ordinals by On.

There are few important things to recognize about ordinals. The empty set $\emptyset$ is an ordinal. Let $\mathcal{A}$ be a non-empty set of ordinals, then $\cap\mathcal{A}$ and $\cup\mathcal{A}$ are ordinals. This is seen by the fact that transitivity and linearity both hold. Furthermore, any intersection of a non-empty class of ordinals is still an ordinal, by the subset axiom. Soon we will see that On itself has a well-ordering structure defined by the $\in$-relation. The union and intersection of a non-empty set of ordinals $\mathcal{A}$ will be the supremum and minimum of $\mathcal{A}$ respectfully. Additionally, the order structure $\leq$ on On is given by subset relation $\subset$. The relation $<$ here then corresponds to the membership relation $\in$.

Any element of a transitive set is transitive. We have two important class functions $\mathcal{P}$ and S, where $\mathcal{P}(A)$ is the power set of A and $S(A) = A \cup \{A\}$. Both take a transitive set to a transitive set. Furthermore, the class function S takes an ordinal to an ordinal.

Theorem 1.50. *Let X be a set. Then following conditions hold:*

1. *It holds that $X \notin X$.*
2. *If X is an ordinal, it is well-ordered.*
3. *Let γ be an ordinal. Denote the closure of γ by $\overline{\gamma}$, which consists of those ordinals that are strict subsets of γ. Then $\gamma = \overline{\gamma}$.*

4. *The class of ordinals is well-ordered by the* $\in$ *relation.*

Proof.

1. If $X \in X$, then the set $\{X\}$ would not contain an element disjoint from it, which contradicts the axioms of regularity.

2. Assume that X is an ordinal. Let $A \subset X$ be non-empty. By the axiom of regularity, there exists $a \in A$ that is disjoint from A. Hence a is the minimum element of A with respect to the $\in$-relation.

3. Let γ be an ordinal. We show that $\gamma = \overline{\gamma}$. Any element of an ordinal is an ordinal by transitivity. Thus $\gamma \subset \overline{\gamma}$.

   For the other direction, define the inductive set I to consists of those elements β of $S(\gamma)$ where $\beta = \overline{\beta}$. Let $\beta \in S(\gamma)$ and assume that $\beta = [0, \beta) \subset I$. We need to show that $\beta \in I$ and in other words $\beta = \overline{\beta}$. Assume that $\alpha \in \overline{\beta}$. Hence $\alpha \subsetneq \beta$. Thus there exists an element θ in β that is not in α. Since α is an initial segment of β, θ is an upper bound on α. If $\alpha = \theta$, then $\alpha \in \beta$. We may assume that $\alpha \neq \theta$ and so $\alpha \subsetneq \theta$. Since $\theta \in I$, we have $\alpha \in \theta \in \beta$ and by transitivity $\alpha \in \beta$. Hence $\overline{\beta} = \beta$ and so $\beta \in I$. Thus $I = S(\gamma)$. Especially $\gamma = \overline{\gamma}$.

4. Now we show that the class of ordinals has an ordinal kind of structure. It suffices to prove linearity and well-ordering. First we are going to show the linearity. Let α and β be ordinals for which linearity doesn't hold. Define $\gamma = \alpha \cap \beta$ and notice that γ is an ordinal. By assumption $\gamma \neq \alpha$ and $\gamma \neq \beta$. Thus $\gamma \in \alpha$ and $\gamma \in \beta$, because $\alpha = \overline{\alpha}$ and $\beta = \overline{\beta}$. Hence $\gamma \in \gamma$, which yields a contradiction. Thus linearity holds for any pair of ordinals.

   For well-ordering, fix any non-empty class $\mathcal{A}$ of ordinals. The intersection of $\mathcal{A}$ is a set, since choosing an element in $\gamma \in \mathcal{A}$, we have that the intersection $\cap\mathcal{A} = \cap\mathcal{B}$, where $\mathcal{B} = \cap\{\gamma \cap \alpha \mid \alpha \in \mathcal{A}\}$. The class $\mathcal{B}$ is a set because of the power set and the subset axioms. Thus $\mathcal{A}$ has a smallest element.

□

Theorem 1.51 (Transfinite induction of ordinals)**.** *Let I be any class of ordinals where for all ordinals α, $\alpha \subset I$ implies that $\alpha \in I$. Then $I = On$.*

Proof. If the statement doesn't hold, then we may define the class J of ordinals for which the claim doesn't hold. Now there exists a minimal ordinal $\alpha \in J$. Thus $\alpha = [0, \alpha) \subset I$. Hence $\alpha \in I$, which is a contradiction. Thus $I = \text{On}$. □

Theorem 1.52 (Transfinite recursion of ordinals)**.** *Let $T : V \to V$ be a class function. Then there exists a unique class function $F : On \to V$ where the recursion*

$$F(\alpha) = T(\alpha, F \mid \alpha)$$

holds for all ordinals α.

Proof. The uniqueness follows from a straightforward induction. Let I be the class of ordinals α, where there exists a unique class function $f = f_\alpha : \alpha \to V$, where $f(\beta) = T(\beta, f \mid \beta)$ for all $\beta < \alpha$. It suffices to show that I is the class of all ordinals, because then we may define $F(\alpha) = f_{\alpha^+}(\alpha)$ for all ordinals α. By the axiom of replacement $F \mid \alpha$ is a set and hence we would see that the recursion would hold for F.

Let $\alpha \subset I$ be an ordinal. We will show that $\alpha \in I$. Assume first that α is a limit ordinal. Then we may define $f_\alpha = \cup_{\beta<\alpha} f_\beta$. Thus f_α is a function, since, if $\beta < \gamma < \alpha$, then $f_\beta = f\gamma \mid \beta$. Furthermore, we see that f_α satisfies the recursion and $\alpha \in I$.

Assume then that α^- exists. Then we define $f_\alpha = f_{\alpha^-} \cup \{(\alpha^-, T(\alpha^-, f \mid \alpha^-))\}$. Hence $\alpha \in I$. This proves the claim. □

We have the following practical corollary:

Corollary 1.53. *Let $T : V \to V$ be a class function. Then there exists a unique class function $F : On \to V$, where*

$$F(\alpha) = \begin{cases} T(F(\alpha^-)), \text{ if } \alpha^- \text{ exists} \\ \cup_{\beta<\alpha} F(\beta), \text{ else.} \end{cases}.$$

Furthermore, if P is a class that contains the empty set, is closed under unions and $T(P) \subset P$, then $F(\alpha) \in P$ for every ordinal α.

Proof. The unique existence of F follows from 1.52. If P is a class that contains the empty set, is closed under unions and is preserved under T, then by a direct induction argument we see that $F(\alpha) \in P$ for all ordinals α. □

Notice that the power set class function $\mathcal{P}$ preserves transitivity, the empty set is transitive and a union of transitive sets is transitive. Similarly the successor class function $S(x) = x \cup \{x\}$ preservers the ordinality of a set, the empty set is an ordinal and a union of ordinals is an ordinal.

Consider Corollary 1.53. If we choose the class function T to be the successor class function, then the identity on the class of ordinals satisfies the recursion equation in Corollary 1.53. Thus we see that every ordinal can be transfinitely obtained from the empty set by applying the successor class function.

Theorem 1.54. *Let X be a well-ordered set. If two strict initial segments of X are isomorphic, then the isomorphism is an identity.*

Proof. Denote with $I \subset X$ the set of elements of $x \in X$, where, if $[0, x) \cong [0, a)$, then the isomorphism is an identity for any $a \in X$. Let $t \in X$ and assume that $[0, t) \subset I$. It suffices to show that $t \in I$. Assume that $f : [0, t) \cong [0, a)$. Since f is an isomorphism, it maps any initial segment of $[0, t)$ to an initial segment of $[0, a)$. Therefore any restriction $f \mid [0, \alpha) : [0, \alpha) \cong [0, f(\alpha))$ is an identity. If x is a limit element, then

$$[0, x) = \cup_{\alpha<x}[0, \alpha),$$

and so the function f must be an identity function. Assume then that x^- exists. Now f must map $[0, x^-]$ to itself. Since the target of f is an initial segment, it must map x to itself. Thus f is an identity. Therefore $t \in I$ and by the transfinite induction principle, $I = X$. □

Corollary 1.55. *Let X and Y be well ordered sets. Then there exists at most one order isomorphism between them.*

Proof. If both $f, g : X \to Y$ are order isomorphisms, then fg^{-1} is an order isomorphism on X. By Theorem 1.54, fg^{-1} is the identity. Thus $f = g$. □

1.5.1 Cardinals

Now we are starting to be able to define the idea of a size of a set called cardinal. They too are ordinals and for a set, the cardinality of the set is defined to be the least ordinal to which there is a bijective correspondence. The first natural question is, is it well-defined, and more specifically formulated, does such an ordinal exist? We answer it by the following theorem:

Theorem 1.56. *Let X be a set, and let $t : \theta \to X$ be injective. Then there exists a bijection $\alpha \to X$ that extends t for some ordinal $\alpha \geq \theta$. Furthermore, if X is a well-ordered set and θ is an increasing map onto an initial segment of X, then the extension $\alpha \to X$ can be chosen to be an order isomorphism.*

Proof. We prove this claim by choosing the elements in order in X, until all elements are chosen. This defines the bijective correspondence with an ordinal and the set X.

We may assume that $\theta \subset X$ and t is a non-surjective inclusion. Denote $\mathcal{A} = \mathcal{P}(X) \setminus \{\emptyset\}$. We apply the axiom of choice to get the choice function $f : \mathcal{A} \to X$, where $f(A) \in A$ for every $A \in \mathcal{A}$.

We define a class function $T : V \to V$ by the following: If $g : \beta \to X$ is a non-surjective function where β is an ordinal, then

$$T(g) = \begin{cases} \min(\theta \setminus \mathrm{im}(g)), \text{ if im } (g) \subsetneq \theta \\ f(X \setminus \mathrm{im}(g)), \text{ else.} \end{cases}$$

For other sets A, we define $T(A) = X$. By transfinite recursion there exists a class function $F : \mathrm{On} \to V$, where $F(\alpha) = T(F \mid \alpha)$. If $F(\alpha) = X$ for some ordinal α, we are done, since by choosing the minimal of such ordinals α we attain a bijection $F \mid \alpha$ which extends θ and especially $\theta \leq \alpha$. Assume that no such α exists. Then the image of F is in X and F is bijective to it's image. Invoking the axiom of replacement on the class function F^{-1}, we see that the class of ordinals On is a set. This cannot be true, since then this set of all ordinals would contain itself as an element.

Lastly, if X is a well-ordered set and, if the map $t : \theta \to X$ is an isomorphism to some initial segment of X, then we may choose the choice function to be the function that takes a set to it's minimum element. By applying the previous argument we attain the order isomorphism. □

As an immediate consequence it follows that every set has a well-ordering structure. Furthermore, if $A \subset X$ has a well-ordering, then it can be extended to X. Additionally, all well-ordered sets are isomorphic to some unique ordinal. Now we are able to define the cardinality of a set as an ordinal.

Definition 1.57. The cardinality of a set X is the least ordinal α where there exists a bijection between X and α. This defines a class function $\mathrm{Card} : V \to \mathrm{On}$ where a set is taken to its cardinality.

From this point onward we consider the set of natural numbers as the first infinite ordinal. Notice that different models of **ZFC** could have different kind of collections as natural numbers, since the concept of an infinite set is an internal concept to each model of set theory.

Lemma 1.58 (Zorn's lemma)**.** *Let $\mathcal{H}$ be a non-empty poset, where every chain is bounded from above. Then $\mathcal{H}$ has a maximal element.*

Proof. Assume for the contradiction that there exists no maximal element in $\mathcal{H}$. Define the set $\mathcal{A}$ to be the set of all linearly ordered subsets of $\mathcal{H}$. Thus, by the axiom of choice we have a function $f : \mathcal{A} \to \mathcal{H}$, to $\mathcal{H}$ that maps a chain A to element $f(A)$. If A is non-empty, then by choice $f(A)$ is a strict upper bound on A. Define a class function $T : V \to V$ where $T(A) = f(A)$, if $A \in \mathcal{A}$ and else A is mapped to the empty set. By Recursion Theorem, there exists a function $F : \mathrm{On} \to V$ where $F(\alpha) = T(F[\alpha]))$, where $F[\alpha]$ is the image of the set α under F. We see that F is injective with the image in $\mathcal{H}$. Hence by the replacement axiom the class of ordinals is a set, which is impossible. □

Lemma 1.59. *The following conditions hold:*

1. *If there exists an injection $A \to B$, then $Card(A) \leq Card(B)$.*
2. *For every set A, it follows that $Card(A) < Card(\mathcal{P}(A))$*
3. *For an infinite set A it holds, that $Card(A \times A) = Card(A)$.*
4. *Let κ be an infinite cardinal. Assume that $Card(A_i) \leq \kappa$ for all $i \in I$ where $Card(I) \leq \kappa$. Then $Card(\cup_{i \in I} A_i) \leq \kappa$.*
5. *The minimum of any class of cardinals is the intersection. The union of a set cardinals is a cardinal.*
6. *Infinite cardinals are limit ordinals.*

Proof.

1. We may assume that $A \subset B$. There is a bijection $t : \theta \to A$ where θ is an ordinal. By Theorem 1.56 this can be extended to a bijection from an ordinal α to B, where $\theta \leq \alpha$. Thus $\mathrm{Card}(A) \leq \mathrm{Card}(B)$.
2. It suffices to show that there exists no surjection $f : A \to \mathcal{P}(A)$. Assume that f is such a surjection. We define then a set B that consists of those points $a \in A$, where $a \notin f(a)$. Since f is a surjection, there exists $a \in A$ where $f(a) = B$. On one hand, if $a \in B$, then $a \notin f(a)$ which cannot be. On the other hand, if $a \notin B$, then $a \in f(a)$, which cannot be either. Thus we reach a contradiction. Thus no such map f exists.

3. Let X be an infinite set. Using Zorn's lemma, we show that $\mathrm{Card}(X \times X) = \mathrm{Card}(X)$. It suffices to show that there exists a surjection $X \to X \times X$. Let $\mathcal{H}$ be the poset defined on those pairs (A, f) where $A \subset X$ is infinite and $f : A \to A \times A$ is a surjection. We declare that $(A, f) \leq (B, g)$, if $A \subset B$ and g is an extension of f.

   We know that $\mathcal{H}$ is non-empty because it contains a copy of natural numbers which do have this property. Consider the map
   $$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
   that maps the n^{th} diagonal to the values $\frac{n(n+1)}{2}, \ldots, \frac{(n+2)(n+1)}{2} - 1$. This defines a bijection. By Zorn's lemma we attain a maximal element in $\mathcal{H}$, since every linearly ordered non-empty chain has an upper bound by taking unions. Denote some maximal element by (A, f). If $\mathrm{Card}(A) = \mathrm{Card}(X)$, then we are ready. Assume that $\mathrm{Card}(A) < \mathrm{Card}(X)$. Then there exists a set $B \subset X$ that is disjoint from A and has the same cardinality as A. Now we have a diagram:
   $$\begin{tikzcd}
   (A \cup B) \times (A \cup B) & (A \times A) \cup (A \times B) \cup (B \times A) \cup (B \times B) \arrow[l, "="'] \\
   & (A \times A) \cup B \arrow[u] \\
   & A \cup B \arrow[u]
   \end{tikzcd}$$
   This function, that is the composition of the two vertical maps in the diagram, is chosen such that it's surjective and extends f. This is possible, since the cardinality of B is equal to the cardinality of $(A \times B) \cup (B \times A) \cup (B \times B)$. This contradicts the maximality. Hence the cardinality of A is the same as the cardinality of X and thus there exists a surjection $X \to X \times X$.

4. Let κ be an infinite cardinal with sets X_i, I having cardinalities at most κ for all $i \in I$. We will show that $\mathrm{Card}(\cup_{i \in I} X_i) \leq \kappa$. We have a surjection from the disjoint union $\sqcup_{i \in I} X_i \to \cup_{i \in I} X_i$. Now
   $$\begin{aligned}
   \mathrm{Card}(\sqcup_{i \in I} X_i) &\leq \mathrm{Card}(\sqcup_{i \in I} \kappa) \\
   &= \mathrm{Card}(I \times \kappa) \\
   &\leq \mathrm{Card}(\kappa \times \kappa) \\
   &= \kappa.
   \end{aligned}$$

5. The minimum case is clear, since ordinals form a well-ordered structure. Let X be a set of cardinals. Then $\cup X$ is the supremum as an ordinal. It suffices to see that $\cup X$ is a cardinal. Let $\alpha \in \cup X$. Now $\alpha < \kappa$ for some $\kappa \in X$ and thus there exists no bijection from α to κ. Hence there exists no bijection from α to $\cup X$. Thus $\cup X$ is a cardinal.

6. An infinite cardinal must be a limit ordinal, since for an infinite ordinal α it holds that α is in bijection with α^+. This bijection exists, since α^+ injects into $\alpha \times \alpha$.

□

1.5.2 Rank

We show what the true power of the regularity axiom is. It defines the cumulative von Neumann hierarchy of sets, which we have denoted by V, the class of all sets. We are able to specify the instant when an arbitrary set is created through transfinite recursion. The rank of a set means the birthday of the set.

By the regularity axiom of $\in$ relation, we have a new form of induction called $\in$-induction.

Theorem 1.60 ($\in$-induction). *Let P be a class with the property that, if $X \subset P$ and X is a set, then $X \in P$. Then P is the class of all sets.*

Proof. Let X be a set and $X \notin P$. Thus for some $x \in X$, $x \notin P$. Hence we may define a sequence $(x_i)_{i\in\mathbb{N}}$, where $x_0 = X$ and $x_{n+1} \in x_n$ for every $n \in \mathbb{N}$. Thus by the axiom of replacement $\{x_n \mid n \in \mathbb{N}\}$ and it contains no element disjoint from it. This contradicts the axiom of regularity. □

Theorem 1.61. *Let $G : On \to V$ be the class function that has the following recursion*

$$G(\alpha) = \begin{cases} \mathcal{P}(\alpha^-), & \textit{if } \alpha^- \textit{ exists,} \\ \cup_{x<\alpha} G(x), & \textit{if } \alpha \textit{ is a limit ordinal.} \end{cases}$$

Then $G(\alpha)$ is transitive for all ordinals α and $P := \cup_{\alpha\in On} G(\alpha) = V$.

Proof. Transitivity of G follows from Corollary 1.53. We use $\in$-induction to show that all sets are included in P. Assume that X is a set and $X \subset P$. We need to show that $X \in P$. Let $x \in X$. Now $x \in P$. Therefore there exists an ordinal β where $x \in G(\beta)$. Choose α_x to be the minimal ordinal, where $x \in G(\alpha_x)$. By the replacement axiom, the class of ordinals $\alpha_x, x \in X$ is a set and therefore the union α is an ordinal. By transitivity, $X \subset G(\alpha)$ and thus $X \in G(\alpha^+)$. Thus P is the class of all sets by $\in$-induction. □

We denote the sets $G(\alpha)$ by V_α. Now we see that $V = \cup_{\alpha\in \mathrm{On}} V_\alpha$. From this we name V the von Neumann cumulative hierarchy of sets.

Definition 1.62. The rank of a set X, $\mathrm{Rank}(X)$, is the least ordinal α such that $X \subset V_\alpha$.

1.5.3 Grothendieck universe

Now we give a definition for different types of ordinals. The most important for us is the inaccessible ordinal, since that gives us what we need for category theory, the Grothendieck universe.

Definition 1.63. Let γ be a limit ordinal.

1. We say that the cofinality of γ is the least ordinal α for which there exists a function $f : \alpha \to \gamma$ where the supremum of the image of f equals γ. We denote the cofinality of γ by $\mathrm{cf}(\gamma)$.

2. We say that γ is a regular ordinal, if $\mathrm{cf}(\gamma) = \gamma$.

3. If $\gamma > \mathrm{Card}(\mathbb{N})$ and γ is a regular ordinal, we say that γ is weakly inaccessible.

4. The limit ordinal γ is said to be inaccessible, if it's weakly inaccessible and for any $\alpha < \gamma$, $\mathrm{Card}(\mathcal{P}(\alpha)) < \gamma$.

5. We say that a set U is a Grothendieck universe, if $U = V_\gamma$ where γ is an inaccessible ordinal.

Notice that the ordinal of natural numbers is a regular cardinal. Similarly for every infinite cardinal, κ, the successor cardinal κ^+ is a regular cardinal.

Notice that, if γ is a limit ordinal, then the identity $\gamma \to \gamma$ is such that the supremum of the image is γ. So the cofinality of a limit ordinal is well-defined and $\mathrm{cf}(\gamma) \leq \gamma$.

Theorem 1.64.

1. *The class function cf becomes an idempotent class function on the limit ordinals, meaning that $cf \circ cf = cf$. Additionally, regular ordinals are cardinals themselves.*

2. *If γ is a regular cardinal and the cardinalities of the sets X_i and I are less than γ for all $i \in I$, then $Card(\cup_{i\in I} X_i) < \gamma$. In other words, if a set S and all it's elements have cardinalities less than γ, so does $\cup S$.*

Proof.

1. Let γ be a limit ordinal. We need to see that $\beta := \mathrm{cf}(\gamma)$ is a limit ordinal and regular. The ordinal β must be a limit ordinal since otherwise we would have a function $f : \beta^- \to \gamma$, whose supremum of the image is γ. Thus β wouldn't be minimal.

   Assume that there exists a function $f : \alpha \to \beta$ where α is an ordinal and sup $\mathrm{im}(f)$ is β. We will show that $\beta \leq \alpha$ and this shows that $\mathrm{cf}(\beta) = \beta$. Since β is the cofinality of γ, there exists a function $g : \beta \to \gamma$, where sup im$(g) = \gamma$. We may assume that g is increasing. Thus the supremum of the image of gf is γ and therefore $\beta \leq \alpha$. Therefore $\beta = \mathrm{cf}(\beta)$.

Lastly, the cardinality of β is β since otherwise there would be a smaller ordinal that is in bijective correspondence with β, but this contradicts the regularity of β.

2. Assume that γ is a regular cardinal and X_i and I are sets where $\mathrm{Card}(I), \mathrm{Card}(X_i) < \gamma$ for $i \in I$. We may assume that I is an ordinal α. Define the map $f : \alpha \to \gamma$, where $f(i) = \mathrm{Card}(X_i)$ for $i \in \alpha$. By regularity the image of f is bounded above by some θ in γ. Set $\beta = \alpha \cup \theta$. Now $\beta < \gamma$. We may assume that β is an infinite ordinal. Then $\mathrm{Card}(\cup_{i \in \alpha_i} X_i) \leq \mathrm{Card}(\beta \times \beta) = \mathrm{Card}(\beta) < \gamma$.

□

Lemma 1.65. *Let X be a set and let γ be an ordinal. Then the following claims hold:*

1. *If $Rank(x) < \gamma$ for all $x \in X$, then $Rank(X) \leq \gamma$. Especially $Rank(X) = sup\{Rank(x) + 1 \mid x \in X\}$ and the rank of an ordinal is the ordinal itself.*

2. *If X is transitive, γ is a regular cardinal and $Card(x) < \gamma$ for all $x \in X$, then $Rank(X) \leq \gamma$.*

Proof.

1. Assuming that the rank of x is less than γ for all $x \in X$, it follows that $x \in V_\gamma$ for all $x \in X$. Thus $X \subset V_\gamma$. Thus the rank of X is at most γ.

2. Assume that X is transitive, γ is regular and the cardinality of each element $x \in X$ is less than γ. Define a class $P = \{A \mid \text{ if } A \in X, \text{ then } \mathrm{Rank}(A) < \gamma\}$. Assume that $A \subset P$ is a set. We will show that $A \in P$. Assume that $A \in X$. By transitivity $A \subset X$ and hence every element of A has cardinality less than γ. By the assumption $A \subset P$, it follows that the ranks of the elements in A are less than γ. Thus we have a function $A \to \gamma$ defined by the rank. Since $A \in X$, $\mathrm{Card}(A) < \gamma$. By regularity of γ it follows that the ranks of the elements of A are bounded from above by some $\theta \in \gamma$. Thus $A \subset V_\theta$ and hence the rank of A is less than γ and hence $A \in P$. Thus P is the class of all sets. Especially, the rank of X is at most γ.

□

Theorem 1.66 (Characterization Theorem)**.** *Let U be a set and let γ be an inaccessible ordinal. Then $U = V_\gamma$, if and only if for every set A it is true that $A \in U$ is equivalent with $A \subset U$ and $Card(A) < \gamma$. Furthermore, $Card(V_\gamma) = \gamma$.*

Proof. Assume that $U = V_\gamma$. Let $A \subset U$ have cardinality less than γ. The rank defines a function $A \to \gamma$. By the regularity of γ, there exists $\theta < \gamma$ where $\mathrm{Rank}(x) < \theta$ for all $x \in A$. Hence $A \subset V_\theta$ and so $A \in \mathcal{P}(V_\theta) \subset V_\gamma$.

Let I be the set that contains those ordinals $\alpha < \gamma$ where $\mathrm{Card}(V_\alpha) < \gamma$. Assume that $\alpha < \gamma$ and $\alpha \subset I$. We need to show that $\alpha \in I$. If α is a successor ordinal, then $V_\alpha = \mathcal{P}(V_{\alpha^-})$ and since γ is inaccessible, $\alpha \in I$. If α is a limit ordinal, then $V_\alpha = \cup_{\beta < \alpha} V_\beta$. Thus by regularity, $\mathrm{Card}(V_\alpha) < \gamma$ and $\alpha \in I$. Therefore $\mathrm{Card}(V_\alpha) < \gamma$ for all $\alpha < \gamma$. Now we see the converse that given any $A \in V_\gamma$, we have $A \subset V_\alpha$ for some α and hence $\mathrm{Card}(A) < \gamma$. Additionally, because V_γ is a union of γ sets where the sets have cardinality less than γ, and $\gamma \subset V_\gamma$; it follows that $\mathrm{Card}(V_\gamma) = \gamma$.

Now suppose for every set A, $A \in U$, if and only if $A \subset U$ and $\mathrm{Card}(A) < \gamma$. First we show $V_\gamma \subset U$. Let I be the set of ordinals $\alpha < \gamma$ where $V_\alpha \subset U$. Assume that $\alpha \subset I$ and $\alpha < \gamma$. Since $\emptyset \in I$ we may assume that $\alpha > \emptyset$. Let $A \in V_\alpha$. Thus $A \subset V_\theta$ for some $\theta < \alpha$. Hence $A \subset U$ and $\mathrm{Card}(A) < \gamma$. Therefore $A \in U$. Hence $V_\alpha \subset U$ and $\alpha \in I$. Therefore $V_\gamma \subset U$.

Lastly, $U \subset V_\gamma$: The set U is transitive since, if $x \in y \in U$, then $y \subset U$ and hence $x \in U$. Since for every element $x \in U$, the cardinality of x is bounded from above strictly by γ, it follows that $\mathrm{Rank}(U) \leq \gamma$ by the second part of Lemma 1.65. In other words $U \subset V_\gamma$. □

Theorem 1.67. *Let U be a set. Then U is a Grothendieck universe, if and only if the following closure properties hold:*

1. *U is transitive (closure under $\in$ relation).*

2. *U is closed under power setting: Let $X \in U$, then $\mathcal{P}(X) \in U$.*

3. *$\mathbb{N} \in U$ (U satisfies the axiom of infinity).*

4. *Closure under replacement and union: Let $f : X \to U$ be a function, where $X \in U$. Then it holds for the image S of f that $\cup S \in U$.*

Furthermore, if U is a Grothendieck universe, then $U = V_\gamma$ where γ is the cardinality of U.

Proof. Assume that U is a Grothendieck universe. Then there exists an inaccessible cardinal γ where $U = V_\gamma = \cup_{\alpha<\gamma} V_\alpha$. The set U is transitive by Theorem 1.61. Let $X \in U$. We see directly that $\mathcal{P}(X) \in U$. Since $\mathbb{N}$ is the least infinite ordinal, $\mathbb{N} \in U$. Assume that $X \in U$ and $f : X \to U$. Denote the image of f by S. Since $S \subset U$ and the elements of S have cardinality less than γ, $\cup S$ has cardinality less than γ by the regularity of γ. Now $S \in U$ and by transitivity $\cup S \subset U$. Hence $\cup S \in U$. Here Lemma 1.66 was used twice.

For the other direction, assume that U satisfies the closure conditions. We need to show that U is a Grothendieck universe. To prove this we use the Characterization Theorem. For this purpose we construct an inaccessible cardinal γ. Let γ be the supremum of the cardinals $\mathrm{Card}(x), x \in U$. First we notice that $\gamma \subset U$: Denote $I = \gamma \cap U$. Assume that $\alpha \in \gamma$ and $\alpha \subset I$. Now, if α^- exists, then $\alpha \subset \mathcal{P}(\alpha^-)$ and so $\alpha \in U$. If α is a limit ordinal, then $\alpha = \cup_{\beta<\alpha}\beta$. Since $\alpha < \gamma$, by definition, there exists a set $A \in U$ with cardinality at least α. Hence we may define a surjection $f : A \to \alpha$. Now $\alpha = \cup(\mathrm{im}(f))$ and since $\alpha \subset U$, it follows that $\alpha \in U$ by the fourth closure condition. Thus $I = \gamma$. Hence $\gamma \subset U$.

For the inaccessibility, notice that γ is a limit ordinal, since it is an infinite set and a supremum of cardinals. Let $\alpha < \gamma$. Now $\alpha \in U$ and $\mathcal{P}(\alpha) \in U$ and so $\mathrm{Card}(\mathcal{P}(\alpha)) < \gamma$. For the regularity, fix a set $\alpha < \gamma$ and a function $f : \alpha \to \gamma$. Thus the image R of f satisfies $\cup R \in U$, by assumption. Now $\cup R \neq \gamma$, since otherwise $\gamma < \gamma$ by the definition of γ. Hence γ is regular and since $\gamma > \mathbb{N}$, γ is an inaccessible cardinal.

If $x \in U$, then $x \subset U$ and $\mathrm{Card}(x) < \gamma$. Assume that $x \subset U$ and $\mathrm{Card}(x) < \gamma$. We have a bijection $\alpha \to x$ for some $\alpha \in U$. If $a \in U$, then $\{a\} \in U$ since $\{a\} \in \mathcal{P}(\mathcal{P}(a)) \in U$. Hence we have function $g : a \to U, b \overset{g}{\mapsto} \{f(b)\}$. The image of g is $\{\{b\} \mid b \in x\}$. Thus $x = \cup(\mathrm{im}(g)) \in U$. By the Characterization Theorem, $U = V_\gamma$. □

Corollary 1.68. *Let U be Grothendieck universe. Then $(U, \in|)$ where $\in|$ is the restriction to U of the binary membership-relation $\in$, is a model for set theory.*

Proof. Axioms of (1) extensionality, (3) subsets, (4) union and (8) regularity are satisfied automatically by any transitive set. By the Characterization Theorem 1.66 and Theorem 1.67, axioms of (2) unordered pair, (5) power set, (6) infinity and (7) replacement are satisfied. Lastly, the axiom of choice holds, since the choice function itself is a set in U, because we code functions as special kind of relations. □

We choose to assume one more thing from our model $(V, \in)$, the axiom of universes: There exists arbitrarily large inaccessible cardinals. In other words for every ordinal γ there exists an inaccessible cardinal κ where $\gamma < \kappa$.

The Grothendieck universes correspond bijectively with inaccessible cardinals and form a well-ordered structure. By the axiom of universes every set exists in some Grothendieck universe. Let U be a Grothendieck universe. We say that an element of U is a $U-$small set and and a subset of U is called a U-moderate set. If a set X is not U-moderate and hence not U-small, we call the set X U-large.

Consider the following statements:

Any set function f has at most one inverse.

Any U-small set function f has at most one inverse for all Grothendieck universes U.

If we were only to assume the existence of a single Grothendieck universe, the latter claim wouldn't necessarily imply the former. For the latter to imply the former, we need to assume that every set is included in some Grothendieck universe. This motivates the axiom of universes.

Since the class of Grothendieck universes is well-ordered, we may choose the smallest Grothendieck universe and do our theory there, but our proofs wouldn't necessarily hold for all sets. Therefore we choose to work in an arbitrary Grothendieck universe. Hence our theory of categories is dependent on which universe we are working in. Our proofs don't depend on the choice of the universe and therefore this point is a semantic one.

Chapter 2

Category theory

A category formalizes the notion of a mathematical structure. First we study the notion of a generalized graph, a multigraph, since every category can be considered an operationally structured multigraph.

2.1 Multigraph

Definition 2.1 (Multigraph)**.** Let V and E be sets and $\mathrm{dom}, \mathrm{cod} : E \to V$ be functions. Then we call the tuple $G := (V, E, \mathrm{dom}, \mathrm{cod})$ a multigraph or a quiver.

The elements of V and E of a multigraph $G = (V, E, \mathrm{dom}, \mathrm{cod})$ are called vertices/objects and edges/arrows correspondingly. The functions dom and cod define the direction of each arrow $f \in E$ starting from domain $\mathrm{dom}(f)$ and ending in codomain $\mathrm{cod}(f)$. We denote this by $x \xrightarrow{f} y$ and $f : x \to y$, where $\mathrm{dom}(f) =: x$ and $\mathrm{cod}(f) =: y$. For vertices $a, b \in V$ we define the hom-set

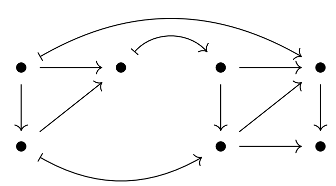
$$G(a, b) := \mathrm{Hom}_G(a, b) := \mathrm{Hom}(a, b) := \{a \xrightarrow{f} b \in E\}.$$

Definition 2.2 (Free multigraph)**.** Let G be a multigraph. A sequence $(f_n, \ldots, f_1)$ of edges of a multigraph G is called a path, if $\mathrm{dom}(f_{i+1}) = \mathrm{cod}(f_i)$ for each $i < n$. We associate an empty sequence for each vertex v and call it the empty path on v for all $v \in V$. The domain of a sequence of morphisms $(f_n, \ldots, f_1)$ in G is defined to be the domain of f_1 and the codomain is defined to be the codomain of f_n. This defines a new multigraph $\mathrm{Free}(G)$ called the free multigraph over G, where the objects are the same as in G and arrows are paths in G.

Definition 2.3 (Multigraph homomorphism)**.** Let G and H be quivers. We say that a pair of functions $F = (f, g)$ is a multigraph homomorphsim from G to H, if f maps the vertices of G to vertices of H and g maps the edges of G to the edges of H such that for any arrow $e : a \to b$ in G it holds that $g(e) : f(a) \to f(b)$. We denote both of the functions f and g by F. Furthermore, we may call a multigraph homomorphism $G \to H$ a diagram of shape G in H.

Notice that there is a canonical embedding of $G \hookrightarrow \mathrm{Free}(G)$ which is an identity on the vertices and maps an edge to its singleton path.

Here's an example of a morphism between graphs defined uniquely by the association of vertices:



When we draw a diagram $f : G \to H$, we usually draw the multigraph G and label each vertex v in G by $f(v)$ and an arrow $e : v \to v'$ in G by $f(e) : f(v) \to f(v')$.

2.2 Category

A category is multigraph with a suitable operation on arrows. In this context we are able to generally define what is meant by commutative diagrams. Furthermore we are able to connect the definition of a category to a first order theory.

Definition 2.4 (Meta Category)**.** Let $\mathrm{Obj}, \mathrm{Mor}$ be sets, let $\mathrm{dom}, \mathrm{cod} : \mathrm{Mor} \to \mathrm{Obj}$ be functions and let $\circ : \mathrm{Mor} \times \mathrm{Mor} \rightharpoondown \mathrm{Mor}$ be a partial function. The tuple $\mathbf{C} = (\mathrm{Obj}, \mathrm{Mor}, \circ, \mathrm{dom}, \mathrm{cod})$ is called a meta category, if the following axioms are satisfied: Let $f, g, h \in \mathrm{Mor}$.

1. The composition $g \circ f$ is defined, if and only if $\mathrm{cod}(f) = \mathrm{dom}(g)$.
2. If the composition $g \circ f$ is defined, then $\mathrm{dom}(g \circ f) = \mathrm{dom}(f)$ and $\mathrm{cod}(g \circ f) = \mathrm{cod}(g)$.
3. If one of the expressions $(h \circ g) \circ f$ and $h \circ (g \circ f)$ is defined, then both are and the expressions are equal.
4. For every $a \in \mathrm{Obj}$ there exist $a \xrightarrow{\alpha} a \in \mathrm{Mor}$ where $\alpha \circ f = f$ and $g \circ \alpha = g$ for all suitable $f, g \in \mathrm{Mor}$.

Let $\mathbf{C} = (\mathrm{Obj}, \mathrm{Mor}, \circ, \mathrm{cod}, \mathrm{dom})$ be a meta category. The elements of $\mathrm{Obj} := \mathrm{Obj}(\mathbf{C})$ and $\mathrm{Mor} =: \mathrm{Mor}(\mathbf{C})$ are called the objects and morphisms of the meta category $\mathbf{C}$, respectively. Since every meta category is defined on a multigraph, we choose to incorporate the language of graph theory in the study of meta categories. The partial function $\circ$ is called the composition of the meta category $\mathbf{C}$. The morphism α in the fourth condition is seen to be unique and hence we shall call it an identity morphism on a and denote it by id_a. The association from objects to morphisms defines an injective map. This allows us to identify objects in $\mathbf{C}$ with the corresponding identity morphisms.

Definition 2.5 (U-category)**.** Let U be a Grothendieck universe. We call a meta category $\mathbf{C}$ a U-category, if the sets of objects and morphisms are U-moderate sets. A U-category $\mathbf{C}$ is called U-locally small, if the hom-set $\mathbf{C}(a, b)$ is U-small for all objects a and b in $\mathbf{C}$. Furthermore, a U-category $\mathbf{C}$ is called U-small, if the sets of morphisms of $\mathbf{C}$ is U-small.

For the later conversation in this thesis we fix a Grothendieck universe U. Further on we leave the universe U implicit and drop mentioning it. In other words we call U-categories, U-locally small categories and U-small categories by categories, locally small categories and small categories, respectively. For every meta category $\mathbf{C}$ there exists a Grothendieck universe with respect to which $\mathbf{C}$ is a small category by the axiom of universes.

The categorical composition symbol " $\circ$ " is sometimes left out of the expressions. For example we will denote gf to mean the same as $g \circ f$. A composite of a path $(f_n, \ldots, f_1)$ in a category $\mathbf{C}$ is the morphism $f_n \ldots f_1$ which is uniquely defined by associativity of composition.

Additionally, a category $\mathbf{C}$ is called connected, if for every pair of objects a, b in $\mathbf{C}$ there exists a sequence of objects $x_i, i = 0, \ldots, n$ where $\mathrm{Hom}(x_i, x_{i+1}) \cup \mathrm{Hom}(x_{i+1}, x_i) \neq \emptyset$ for all $i < n$ and $x_1 = a$ and $x_n = b$.

Example 2.6. Consider the following examples of categories:

1. Category **Set** of sets:
   (a) The set of objects is the Grothendieck universe U.
   (b) Morphisms are functions $f : X \to Y$ between small sets, where the domain and codomain are seen from the notation.
   (c) The categorical composition of morphisms comes from the usual composition of functions.[1]
2. Let L be an alphabet that is a small set and T an L-theory.[2] Define a category $\mathbf{Model}_L^T$ of L, T-models as follows:

[1] Later on, if the category's composition is the function composition, we will leave it as implicit.

[2] The collection of symbols of L can is finite and therefore we may code the symbol for left bracket as some natural number for instance. Thus it is reasonable to think that L is a set.

(a) Objects are small L, T-models $\mathcal{M}$. In other words the universe of the L-model $\mathcal{M}$ is small and $\mathcal{M} \vDash T$.

(b) Morphisms are the L-model morphisms.

This definition contains the categories of sets, groups, monoids, R-modules as examples. Similarly, for posets, prosets and linearly ordered sets. If the theory T is an algebraic or a positive L-theory, then we call the category $\mathbf{Model}_L^T$ an algebraic or a positive category, respectively.

3. Let $P = (X, \leq)$ be a pre-ordered set. Then we get a pre-order category $\mathbf{P}$:

   (a) The set of objects is X.

   (b) The set of morphisms is the relation $\leq$ itself or, if one wishes the graph $\{(a, b) \in X \times X \mid a \leq b\}$ of the relation $\leq$. The domain and codomain of $(a, b) \in \leq$ is a and b, respectively.

   (c) The composition is uniquely defined.

4. Every set X will be thought of as a meta category that we obtain by associating for every element in X a unique identity element.

5. There is the category of **Multigraph** of multigraphs:

   (a) The objects are multigraphs.

   (b) The morphisms are multigraph homomorphisms.

6. Every small monoid is a one object small category, where the composition is the monoid operation.

7. Let $G = (V, E, \mathrm{dom}, \mathrm{cod})$ be a multigraph. Then the free multigraph $\mathrm{Free}(G)$ has a natural categorical structure via concatenation.

   (a) The set of objects is V.

   (b) Morphisms are paths in G.

   (c) Composition is defined by concatenating suitable sequences. Hence

   $$(f_n, ..., f_1) \circ (g_m, \dots, g_1) = (f_n, \dots, f_1, g_m, \dots, g_1)$$

   when $\mathrm{dom}(f_1) = \mathrm{cod}(g_m)$.

   The definition yields a category, because empty sequences become the identity elements and associativity is clear. A category $\mathbf{C}$ is called a free category, if there exists a multigraph G such that $\mathrm{Free}(G) = \mathbf{C}$.

2.3 Duality

Every statement has a dual form in the language of category theory. Proving statements for an arbitrary category always proves the dual statement too. Duality becomes early on a powerful tool.

Definition 2.7. Let $\mathbf{C} = (O, M, \circ, \mathrm{dom}, \mathrm{cod})$ be a category. Define

$$\mathbf{C}^{op} = (O, M, \circ', \mathrm{cod}, \mathrm{dom})$$

where $f \circ' g := g \circ f$, when $g \circ f$ is defined in $\mathbf{C}$. The structure $\mathbf{C}^{op}$ is a category and it is called the dual and opposite category of $\mathbf{C}$. Let f be a morphism of $\mathbf{C}$. When we consider the morphisms f in $\mathbf{C}^{op}$, we denote f by f^{op}.

The dual category consists of the exact same information as the original but the directions of the arrows flip and the composition is defined in the simplest way that one can in such a generality. Notice that the dual of a dual is the original. Even though opposite category contains the same information as the category itself, they can be quite different from category theoretical point of view. For example the opposite category of **Set** looks like the category of complete atomic Boolean algebras [5].

A statements about arbitrary categories have a dual form. If a statement is proved in an arbitrary category $\mathbf{C}$, then the statement holds also in $\mathbf{C}^{op}$. Hence we attain a possibly new true statement in $\mathbf{C}$ and we call this possibly new statement a dual statement. Therefore many proofs give us two different results. Additionally, definitions in an arbitrary category yield a dual concept. It happens that the dual concept of injection is surjection. There is a duality between supremum and infimum and disjoint union and Cartesian product.

2.4 Functors and diagrams

Consider the alphabet $L = (d^1, c^1, R^3, P^1)$, where d, c are function symbols of order 1 and R, P are relation symbols of order 3 and 1, respectively. Since the objects of a category $\mathbf{C}$ can be identified with the associated identity morphisms of $\mathbf{C}$, we may regard the category $\mathbf{C}$ as an L-model, where the relation symbols R and P are interpreted as the composition and the set of identities, respectively. Additionally, the function symbols d and c are interpreted as the domain and codomain functions, respectively.

We may define an L-theory T such that a small category $\mathbf{C}$, with objects identified to the corresponding identities, is an L, T-model and furthermore such that any L, T-model defines canonically a category. This theory T exists, since the axioms of a category can be formulated by sentences of first order predicate logic and the requirement that the interpretation of R is partial function can be formulated as a sentence. We call the alphabet L the vocabulary of categories and the L-theory T the theory of categories. The morphisms among categories are called functors and we may define them through this equivalation as L-model morphisms. To be precise we give the following definition:

Definition 2.8. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A functor $F : \mathbf{C} \to \mathbf{D}$ consists of a pair of function

$$\begin{cases} \mathrm{Obj}(\mathbf{C}) \xrightarrow{F} \mathrm{Obj}(\mathbf{D}) \\ \mathrm{Mor}(\mathbf{C}) \xrightarrow{F} \mathrm{Mor}(\mathbf{D}) \end{cases}$$

such that the diagrams

$$\begin{tikzcd} \mathrm{Mor}(\mathbf{C}) \arrow[r, "F"] \arrow[d, "\mathrm{dom}"] & \mathrm{Mor}(\mathbf{D}) \arrow[d, "\mathrm{dom}"] \\ \mathrm{Obj}(\mathbf{C}) \arrow[r, "F"] & \mathrm{Obj}(\mathbf{D}) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mor}(\mathbf{C}) \arrow[r, "F"] \arrow[d, "\mathrm{cod}"] & \mathrm{Mor}(\mathbf{D}) \arrow[d, "\mathrm{cod}"] \\ \mathrm{Obj}(\mathbf{C}) \arrow[r, "F"] & \mathrm{Obj}(\mathbf{D}) \end{tikzcd} \qquad \begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[rr, bend right, "F(gf)"'] & Fb \arrow[r, "Fg"] & Fc \\ Fd \arrow[rr, "id_{Fd}"] \arrow[rr, bend right, "F(id_d)"'] & & Fd \end{tikzcd}$$

commute for all morphisms $a \xrightarrow{f} b \xrightarrow{g} c$ and objects d in $\mathbf{C}$. We call a functor $\mathbf{C}^{op} \to \mathbf{D}$ a contravariant functor $\mathbf{C} \to \mathbf{D}$. In contrast to contravariant functors, a functor is also called a covariant functor.

The Functor composition is defined via the usual way. Every functor $F : \mathbf{C} \to \mathbf{D}$ defines a functor $F^{op} : \mathbf{C}^{op} \to \mathbf{D}^{op}$, where a morphism f^{op} is taken to $F(f)^{op}$. This uniquely specifies the functor F^{op} and $(F^{op})^{op} = F$. A functor with a small category as its domain is called a diagram.

Definition 2.9. Define a U^+-category **M-CAT** of all categories as follows:

1. Objects are categories with respect to the universe U.
2. Morphisms are functors $F : \mathbf{C} \to \mathbf{D}$.

Similarly we have a categorical structure on locally small categories and small categories. We denote these categories by **CAT** and **Cat**, respectively.

Every small ordinal X is a well-ordered set and hence has a categorical poset structure. We denote the category defined by X with $\mathbf{X}$. Notice that we have identified the collection of natural numbers with the first infinite ordinal. So every natural number has a poset structure.

We use graphs like $\bullet \rightrightarrows \bullet$ to denote the free category it generates. The previous picture means a graph of two vertices with two similarly pointed arrows from one vertex to the other.

Example 2.10.

1. Objects: Let c be an object in a category $\mathbf{C}$. We may identify c with a functor $\bar{c} : \mathbf{1} \to \mathbf{C}, 0 \stackrel{\bar{c}}{\mapsto} c$.

2. Morphisms: Every morphism f in a category $\mathbf{C}$ corresponds to a functor $\overline{f} : \mathbf{2} \to \mathbf{C}$, where the non-trivial arrow of $\mathbf{2}$ is taken to f. The functor $\overline{f}$ is called the diagram defined by the morphism f.

3. Quivers: Every quiver $(V; E; t, s : E \to V)$ corresponds to a functor from $\bullet \rightrightarrows \bullet$ to $\mathbf{Set}$.

4. Monoid homomorphisms: Given monoids M, N and a function $f : M \to N$, it follows that f is a monoid morphism if and only if it is a functor.

5. Given a monoid M, any functor $F : M \to \mathbf{Set}$ correspond to a monoid action by exponential transposition. This means is that the functor F chooses a set X and a monoid homomorphism $M \to \mathbf{Set}(X, X)$, which corresponds exactly to an action $\alpha : M \times X \to X$ on X.

6. Increasing maps: Given two prosets Q and P, any map between them is increasing if and only if it is a functor between the proset categories.

7. Power set: Taking power sets defines two different functors, $\mathcal{P}_*, \mathcal{P}^* : \mathbf{Set} \to \mathbf{Set}$, where the former is covariant and the latter is contravariant. Both take a small set X to its power set
$$\mathcal{P}(X) = \{A \mid A \subset X\}.$$
Given a function $X \xrightarrow{f} Y$ between small sets X and Y, we define
$$\mathcal{P}_* : \mathcal{P}(X) \to \mathcal{P}(Y), \mathcal{P}_*(f)(A) = f[A] \text{ for all } A \subset X$$
and
$$\mathcal{P}^* : \mathcal{P}(Y) \to \mathcal{P}(X), \mathcal{P}^*(f)(B) = f^{-1}[B] \text{ for all } B \subset Y.$$
Notice that, since the direct image and pre-image functions, denoted by f_* and f^*, are increasing with respect to the usual poset structure induced by subset relation, the power set functors also become **Poset** valued functors.

8. For example, the functor
$$\mathbf{Top} \to \mathbf{Set}, (X, \tau) \mapsto X, f \mapsto f$$
is a forgetful functor that forgets the topological structure τ of the space (X, τ). There is a functor
$$\mathbf{Grp} \to \mathbf{Mon}, G \mapsto G, f \mapsto f$$
from the category of groups to the category of monoids, that doesn't lose the information of the group, but it embeds groups into a larger context.

9. Given a locally small category $\mathbf{C}$ and an object a in $\mathbf{C}$, we have two functors $\mathbf{C} \to \mathbf{Set}$, $\mathbf{C}_a$ and $\mathbf{C}^a$, where the first is a covariant functor and latter contravariant:
$$\mathbf{C}_a : \mathbf{C} \to \mathbf{Set}, \begin{cases} b \stackrel{\mathbf{C}_a}{\longmapsto} \mathbf{C}(a, b), \\ \begin{tikzcd} a \arrow[d, "g"'] \arrow[dr, bend left, "(\mathbf{C}_a(f))(g)=fg"] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \end{cases}$$
and
$$\mathbf{C}^a : \mathbf{C}^{op} \to \mathbf{Set}, \begin{cases} b \stackrel{\mathbf{C}^a}{\longmapsto} \mathbf{C}(b, a), \\ \begin{tikzcd} a & \\ b' \arrow[u, "g"] & b \arrow[l, "f"] \arrow[ul, bend right, "(\mathbf{C}^a(f^{op}))(g)=gf"'] \end{tikzcd} \end{cases}$$
We shall denote
$$f_{*,a} = f_* = \mathbf{C}_a(f) \text{ and } f^{*,a} = f^* = \mathbf{C}^a(f^{op}).$$
it is worth noticing that $(\mathbf{C}^{op})_a = \mathbf{C}^a$.

Multigraphs and categories are closely related. The following theorem makes this connection clearer:

Theorem 2.11. *Let $\mathbf{C}$ be a category, G a quiver and $f : G \to \mathbf{C}$ a multigraph homomorphism. Let $i : G \to \mathrm{Free}(G)$ be the canonical embedding. Then there exists a unique functor $F : \mathrm{Free}(G) \to \mathbf{C}$ that extends f in the sense that $F \circ i = f$.*

Proof. If such a functor exists, it is a unique, since it is fixed on objects, a sequence of morphism $(g_n, \dots, g_1)$ must be mapped to the composite of $(f(g_n) \dots f(g_1))$ and the empty sequences are mapped to the corresponding identities.

The pair of functions defined by the previous considerations defines a functor $\mathrm{Free}(G) \to \mathbf{C}$, where $Fi = f$. □

Notice that the correspondence of functors from $\mathrm{Free}(G)$ and multigraph homomorphism from G to a category $\mathbf{C}$ is a canonical bijection. Hence we may identify the two by keeping in mind this bijection where a functor $F : \mathrm{Free}(G) \to \mathbf{C}$ is taken to $F \circ i : G \to \mathbf{C}$.

Definition 2.12 (Commutative diagram)**.** Let $\mathbf{I}$ and $\mathbf{C}$ be categories. Assume that $\mathbf{I}$ is a small category. A functor $F : \mathbf{I} \to \mathbf{C}$ is called a diagram in $\mathbf{C}$ of shape $\mathbf{I}$. If $\mathbf{I}$ is a free category, then F is called a strict diagram. The diagram F is said to commute, if the restriction $F : \mathbf{I}(a, b) \to \mathbf{C}(F(a), F(b))$ is a constant maps for all objects a and b in $\mathbf{I}$.

Almost always we are only interested in the commutativity of a strict diagram. Given a multigraph G and a category $\mathbf{C}$, we define the information of a diagram of shape $\mathrm{Free}(G)$ in $\mathbf{C}$ through a multigraph homomorphism $f : G \to \mathbf{C}$. The commutativity of the diagram $F : \mathrm{Free}(G) \to \mathbf{C}$ corresponding to f is equivalent to f taking all paths $(g_n, \dots, g_1) : a \to b$ to the fixed arrow $f(g_n) \dots f(g_1) : f(a) \to f(b)$ in $\mathbf{C}$. Therefore this definition of commutative diagram agrees with our previous notions of commutative diagrams.

A diagram $F : \mathbf{I} \to \mathbf{C}$ commutes, if and only if there exists a proset category $\mathbf{P}$ through which F factors, meaning that there exist functors $G : \mathbf{I} \to \mathbf{P}$ and $H : \mathbf{P} \to \mathbf{C}$ where the diagram

$$\begin{tikzcd} \mathbf{I} \arrow[rr, "F"] \arrow[dr, "G"'] & & \mathbf{C} \\ & \mathbf{P} \arrow[ur, "H"'] & \end{tikzcd}$$

commutes. The converse is clear. To produce a proset category, it suffices to define a new category $\mathbf{P}$ over the objects in $\mathbf{C}$ and declaring that $\mathbf{P}(a, b)$ consists of a unique element, if $\mathbf{C}(a, b)$ contains an element and $\mathbf{P}(a, b)$ is empty otherwise. We set the morphisms of $\mathbf{P}$ to be the disjoint union of these hom-sets. The composition is then unique and well-defined.

Notice the three special cases about commutative diagrams: If the diagrams

$$\begin{tikzcd} c \arrow[loop above, "f"] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] & b \end{tikzcd} \qquad \begin{tikzcd} & z \arrow[dl, "s"'] \\ x \arrow[r, "p"'] & y \arrow[u, "r"'] \end{tikzcd}$$

commute, then $f = id_c, g = h$ and $srp = id_x$. The diagram

$$\begin{tikzcd} a \arrow[r] \arrow[d] & b \\ c & \end{tikzcd}$$

always commutes.

Definition 2.13 (Lift)**.** Let $\mathbf{C}$ be a category with a diagram

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$$

We say that the morphism g lifts f, if there exists a morphism $t : a \to b$ where the diagram

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[ur, "t"] \arrow[r, "f"'] & c \end{tikzcd}$$

commutes. Additionally, we call t a lift of f along g.

We are able to define a category where lifts are the morphisms.

Definition 2.14 (Slice -, coslice -and arrow category)**.** Let a be an object in a category $\mathbf{C}$.

1. The slice category of $\mathbf{C}$ over a, denoted by $\mathbf{C}/a$, is defined as follows:
   (a) The objects are morphisms f of $\mathbf{C}$ whose codomain is a.
   (b) A morphism $f \to g$ between objects of $\mathbf{C}/a$ is a lift of f along g.
   (c) The composition is the normal composition of morphisms in $\mathbf{C}$.
2. Dually we define the coslice category of $\mathbf{C}$ under a as $(\mathbf{C}^{op}/a)^{op}$. To be specific:
   (a) An object in $a/\mathbf{C}$ is a morphism in $\mathbf{C}$ with a as the domain.
   (b) Let $f : a \to x$ and $g : a \to y$ be objects in $a/\mathbf{C}$. A morphism $\alpha : f \to g$ is a morphism $\alpha : x \to y$ in $\mathbf{C}$ where the following diagram

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"] & y \\ x \arrow[ur, "\alpha"'] & \end{tikzcd}$$

   commutes. The morphism α is called a colift of g along f.
   (c) The composition is just the usual composition in $\mathbf{C}$.
3. Similarly, we define the arrow category $\mathrm{Ar}(\mathbf{C})$ of $\mathbf{C}$ where the objects are morphisms $f : c \to d$ in $\mathbf{C}$ and a morphism $\alpha : f \to g$ in $\mathrm{Ar}(\mathbf{C})$ is a pair (α_1, α_2) of morphisms between the corresponding domains and codomains such that the induced square

$$\begin{tikzcd} f \arrow[d, "\alpha"] \\ g \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\alpha_1"] & c' \arrow[d, "\alpha_2"] \\ d \arrow[r, "g"] & d' \end{tikzcd}$$

   commutes.

Example 2.15.

1. Let 1 be a singleton set. We call the coslice category $1/\mathbf{Set}$, also denoted by $\mathbf{Set}_*$, the category of pointed sets. Notice that the objects of this category can be thought as pairs (X, x), where X is a small set with an element x. The morphisms are then function between the sets that respect the choice of the elements.
2. Let 1 be a singleton topological space. The category of pointed spaces is $1/\mathbf{Top}$, which is also denoted by $\mathbf{Top}_*$.

In the case of categories we add a criterion for liftings:

Definition 2.16 (Diagrammatic lifting problem)**.** Consider a commutative diagram

$$\begin{tikzcd} & \mathbf{C} \arrow[d, "F"] \\ \mathbf{I} \arrow[r, "D"'] & \mathbf{D} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Obj}(\mathbf{C}) \arrow[d, "F"] \\ \mathrm{Obj}(\mathbf{I}) \arrow[ur, "D'"] \arrow[r, "D"'] & \mathrm{Obj}(\mathbf{D}) \end{tikzcd}$$

of categories, where D is a diagram. We call this commutative diagram a diagrammatic lifting problem for F. A lift for D that extends D' to a functor is called solution to the diagrammatic lifting problem. We may denote the lifting problem with the pair (D, D').

As an example of a diagrammatic lifting problem, consider the strict diagram

$$\begin{tikzcd} Fa \arrow[r, "\alpha"] \arrow[d, "f"] & Fb \arrow[d, "g"] \\ Fc \arrow[r, "\beta"'] & Fd \end{tikzcd}$$

in $\mathbf{D}$ where $F : \mathbf{C} \to \mathbf{D}$. The solution that we are looking for is then a diagram

$$\begin{tikzcd} a \arrow[r, "\alpha'"] \arrow[d, "f'"] & b \arrow[d, "g'"] \\ c \arrow[r, "\beta'"'] & d \end{tikzcd}$$

in $\mathbf{C}$ where the morphisms x' map to x under F for $x \in \{\alpha, \beta, f, g\}$. Motivated by lifting problems, we formulate the following definitions:

Definition 2.17. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. The functor F defines a map

$$\mathbf{C}(a, b) \xrightarrow{F_{(a,b)}} \mathbf{D}(F(a), F(b))$$

for every pair of objects a and b in $\mathbf{C}$. If the map $F_{(a,b)}$ is injective for every object a and b in $\mathbf{C}$, then we call the functor F faithful. If the map $F_{(a,b)}$ is surjective for all objects a and b in $\mathbf{C}$, then the functor F is called full. A full and faithful functor is called fully faithful.

Consider the vocabulary L and theory T of categories. Any full functor between small categories, when considered as a morphism in $\mathbf{Model}_L^T$, is a globally full model morphism.

The following theorem gives a nice way to characterize faithfulness and fullness of a functor by diagrammatic lifting problems.

Theorem 2.18. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor, let $D : \boldsymbol{I} \to \boldsymbol{D}$ be a diagram and let $D' : Obj(\boldsymbol{I}) \to Obj(\boldsymbol{C})$ be a function. Assume that the diagram defining the lifting problem commutes.*

$$\begin{tikzcd} & Obj(\boldsymbol{C}) \arrow[d, "F"] \\ Obj(\boldsymbol{I}) \arrow[ur, "D'"] \arrow[r, "D"'] & Obj(\boldsymbol{D}) \end{tikzcd}$$

1. *If F is faithful, then the lifting problem has at most one solution D'. If a solution D' exists and D commutes, so does D'. The converse holds in a strong sense: If all lifting problems defined by a diagram of a single morphism have at most one solution, then F is faithful.*

2. *If F is full and $\boldsymbol{I}$ is a free category, then the lifting problem has a solution. Additionally, F is full, if all lifting problems defined by a single morphism have a solution.*

3. *The functor F is fully faithful, if and only if every lifting problem has exactly one solution.*

Proof.

1. For the converse assume that every diagram $F(a) \xrightarrow{g} F(b)$ in $\mathbf{D}$ has at most one solution. This is just an other formulation that F is injective on hom-sets. Thus F is faithful.

   Assume that F is faithful and assume that D' can be extended to a functor such that the diagram

$$\begin{tikzcd} & \mathbf{C} \arrow[d, "F"] \\ \mathbf{I} \arrow[ur, "D'"] \arrow[r, "D"'] & \mathbf{D} \end{tikzcd}$$

   commutes. Fix a morphism $f : i \to j$ in $\mathbf{I}$. We know that $FD'f : Di \to Dj$ must equal Df. The faithfulness of F implies that there exists at most one morphism $g : D'i \to D'j$ such that $Fg = Df$. Hence $D'f = g$ and so D' is uniquely defined. Furthermore, if D commutes, this means that restriction of D to $\mathbf{I}(i,j)$ would be a constant map and thus

$$\mathbf{I}(i,j) \xrightarrow{D'} \mathbf{C}(D'i, D'j) \xrightarrow{F} \mathbf{D}(Di, Dj)$$

would be a constant map. Due to the faithfulness of F, it would follow that D' is a constant map $\mathbf{I}(i,j) \to \mathbf{C}(D'i, D'j)$ for all objects i and j in $\mathbf{I}$. Hence D' would commute.

2. The converse clearly implies the fullness of F. Assume that D is full and $\mathbf{I}$ is a free category. Let G be a multigraph that generates $\mathbf{I}$. Now every edge $f : i \to j$ in G is mapped to $Df : FD'i \to FD'j$. Since F is full there exists $D'f : D'i \to D'j$. Hence we obtain a multigraph homomorphism $D' : G \to \mathbf{C}$. Now $D' : G \to \mathbf{C}$ extends uniquely to a functor $D' : \mathbf{I} \to \mathbf{C}$ by Theorem 2.11. Since FD' and D agree on the multigraph G, it follows that $FD' = D$.

3. The converse follows from parts 1 and 2. Assume that F is fully faithful. Fix a morphism $f : i \to j$ in $\mathbf{I}$. Define $D'f$ to be the morphism $D'i \to D'J$ that maps to Df under F. Now D' maps identities to identities and by the faithfulness of F. By the functoriality of D, the functor D' respects composition. Thus D' is a functor and a solution to the lifting problem. By faithfulness of F, D' is the unique solution.

□

2.5 Objects and morphisms

In this section we are going to consider properties of objects and morphisms. We generalize the concepts of a homeomorphism in topology and a linear isomorphism in linear algebra. One of the basic conceptual tools in category theory is the concept of a universal property. Universal properties define objects up to a unique isomorphism which suffices as definition of an object to us, even though an object may not be uniquely defined by a universal property. The concept of a universal property is closely related to the notions of initiality and terminality.

2.5.1 Objects and morphisms

A common way we specify the properties of an object is how the morphisms arriving or leaving behave.

Definition 2.19. Let c be an object in a category $\mathbf{C}$. The object c is called initial, if for any object d in $\mathbf{C}$, there exists a unique morphism $c \xrightarrow{!} d$. If c is initial in $\mathbf{C}^{op}$, we call c a terminal object in $\mathbf{C}$. An object is a zero object, if it is both initial and terminal. Those categories that have a zero object are called pointed categories.

Initiality and terminality are dual concepts for each other. An object c is terminal, if and only if for every object d there exists a unique morphism $d \xrightarrow{!} c$.

Example 2.20.

1. The empty set and any one-element set are, respectively, initial and terminal objects in the category **Set** of sets. By Lemma 1.34 all positive categories have a terminal object, whose universe is a singleton set. By the same lemma any algebraic category, with no constant symbols, has an initial object as the empty model.

2. Any one-element monoid is a zero object in the category **Mon** of monoids.

3. The categories of pointed sets and pointed spaces has a zero object.

Definition 2.21 (Iso-,epi-,monomorphisms, retraction and section)**.** Let $f : a \to b$ be a morphism in a category $\mathbf{C}$. We call the morphism f

1. an isomorphism, if there exists $g : b \to a$, where $fg = id_b$ and $gf = id_a$. The morphism g is called the inverse of f and denoted by $f^{-1} := g$. Additionally, if there exists an isomorphism $g : a \to b$, then we say that a and b are isomorphic and denote $g : a \cong b$. A category where every morphism is an isomorphism is called a groupoid.

2. a retraction, if there exists $g : b \to a$ such that $fg = id_b$. Here f is called a left inverse of g and g a right inverse of f.

3. a section, if f is a retraction in $\mathbf{C}^{op}$.

4. an epimorphism, if for any $y, y' : b \to c$, where $yf = y'f$, $y = y'$. An epimorphism f is also called epic.

5. a monomorphism, if it is an epimorphism in $\mathbf{C}^{op}$. The word monic is also used.

Example 2.22.

1. In the category **Set** of sets, the surjections are exactly the epimorphisms and the retractions. The axiom of choice restricted to universe U is equivalent to epimorphisms being retractions in **Set**. Similarly, injections are precisely the monomorphisms and, with non-empty domain, sections. Seeing that injections, with non-empty domain, are sections does not require the axiom of choice.

2. Bijections in **Set** are the isomorphisms. A bijection is by definition a surjection and an injection. Thus in **Set**, if a morphism is both epic and monic, it is an isomorphism. This does not hold in general. A classical counter example is the ring homomorphism, that injects the integers $\mathbb{Z}$ to the rationals $\mathbb{Q}$. In **Top** the exponential map defines a continuous bijection from the interval $[0, 1)$ to the sphere $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. This bijection cannot be an isomorphism in **Top**, since only the space $\mathbb{S}^1$ is compact.

3. Every monic in categories **Mon**, **Grp** and **R-Mod** is injective. This is seen by the fact that the previously mentioned categories have an object c such that a morphism $c \to d$ corresponds canonically with an element of d. The object c can be chosen as $\mathbb{N}, \mathbb{Z}$ and R in categories **Mon**, **Grp** and **R-Mod**, respectively.

4. In the category of R-modules, every epimorphism is surjective. To see this fix an R-linear epimorphism $f : M \to N$. Now $\operatorname{im}(f)$ is a submodule of N and we attain the quotient module $N/\operatorname{im}(f)$.[3] The quotient and zero maps $N \to N/\operatorname{im}(f)$ agree on the image of f and by the epicness of f they must be equal. Since the quotient is trivial, f is surjective.

5. The epimorphisms in **Grp** are seen to be surjections by the following argument:

   Assume $f : G \to H$ is a non-surjective group epimorphism. Then $\operatorname{im}(f)$ is a subgroup of H and it defines a partition of H, $H/\operatorname{im}(f)$, to left cosets. If $\operatorname{im}(f)$ is a normal subgroup of H, it follows that the quotient homomorphism $H \to H/\operatorname{im}(f)$ must equal the zero map by the epicness of f. Thus f would be a surjection, which would contradict the assumption. Hence $|H/\operatorname{im}(f)| > 2$. There are three disjoint left cosets $\operatorname{im}(f)$, $h\operatorname{im}(f)$ and $h'\operatorname{im}(f)$ of H. Define an element σ of the symmetric group S_H as a bijection that swaps the elements of $h\operatorname{im}(f)$ and $h'\operatorname{im}(f)$ by $ha \mapsto h'a \mapsto ha$ for $a \in \operatorname{im}(f)$. The other elements are kept in place by σ. Define

$$\begin{aligned}
&y, y' : H \to S_H,\\
&y(p)(q) = qp^{-1} \text{ and}\\
&y'(p) = \sigma^{-1} \circ y(p) \circ \sigma, p, q \in H.
\end{aligned}$$

   The functions y and y' are homomorphisms. We shall show that $y'f = yf$, but $y \neq y'$. It holds that $y'f = yf$: Let $t \in H$. Now $t = ha$ or $t = h'a$ for some $a \in \operatorname{im}(f)$ or neither. In each of the separate cases we have

$$y'(f(x))(t) = \sigma^{-1}(\sigma(t)f(x)^{-1}) = tf(x)^{-1} = y(f(x))(t).$$

   So $y'f = yf$. Furthermore, $y \neq y'$, since $y(h^{-1})(e) = h$ but

$$y'(h^{-1})(e) = \sigma(\sigma(e)h) = \sigma(h) = h'.$$

[3] If A is a submodule of B, then we are able to define a model congruence on B by $a \sim b$, if $(-b) + a \in A$. The quotient is then denoted B/A. More generally, if G is a subgroup of H, we are able to define the same equivalence relation, but it may not be a congruence. If it is a congruence, we call the subgroup G normal.

Theorem 2.23. *The following statements hold for a morphism $f : a \to b$ in a category $\mathbf{C}$:*

1. *There exists at most one inverse for f.*
2. *If f is a retraction, it is an epimorphism. Dually, f is monic, if it is a section.*
3. *If f is a retraction and monic, then f is an isomorphism. Dually, if f is a section and epic, then f is an isomorphism.*
4. *Let the sets M, E, S, R, I denote the sets of monomorphisms, epimorphisms, sections, retractions and isomorphisms, respectively. Then the sets are closed under composition and for the classes M and S, if a composite gf belongs to either of the classes, then f belongs to the same one. Dually, if a composite morphism gf belongs to one of the classes E and R, then g belongs to the same class.*

Proof.

1. If g and g' are inverses of f, then
$$g = g(fg') = (gf)g' = g'.$$
2. Assume f is a retraction and $yf = y'f$ for $y, y' : b \to c$ where c is an object in $\mathbf{C}$. Then
$$y = yfg = y'fg = y',$$
where g is a right inverse of f. The dual statement is attained by applying the previous for f in the dual category $\mathbf{C}^{op}$.
3. Assume that f is a retraction and monic. We will show that f is an isomorphism. There exists a right inverse g for f. Now $fg = id_b$ and thus $fgf = f = fid_a$. Hence $gf = id_a$, since $gf, id_a : a \to a$.
4. To show the closure under composition, it requires to only check that M and S are closed under composition, since the cases relating to the classes E and R are formally dual and $I = S \cap E$. Let g and f be morphisms, where the composite gf is defined. First assume that g and f are monomorphisms. To show that gf is monic, fix a parallel pair of morphisms x, x' pointing to the domain of f, where $gfx = gfx'$.[4] By monicness of g and f, $fx = fx'$ and $x = x'$. Assume next that g and f are sections. There exist left inverses g' and f', respectively. Now $(f'g')(gf) = f'idf = f'f = id$. Hence gf is a section.

   For the second part assume that gf is monic. Now, if $fx = fx'$ for a parallel pair of morphisms x, x' pointing to the domain of f, then $gfx = gfx'$ and hence $x = x'$. Lastly, if gf is a section, it has a left inverse h, where $h(gf) = id$. Hence $(hg)f = id$ and thus hg is a left inverse of f. Thus f is a section.

□

Theorem 2.24. *Let a and a' be initial objects (or terminal objects) in a category $\mathbf{C}$. Then they are isomorphic via a unique morphism.*

Proof. There exist unique morphisms $f : a \to a'$ and $g : a' \to a$ by the initiality of the objects. Now $gf : a \to a$ and by the initiality of a it holds that $gf = id_a$. Similarly, $fg = id_{a'}$. □

Definition 2.25 (Preserve, reflect and create)**.** Let $F : \mathbf{C}_1 \to \mathbf{C}_2$ be a functor and $P = (P_1, P_2)$ where P_i is a subset of morphisms of $\mathbf{C}_i$ for $i = 1, 2$.

1. We say that F preserves P, if $F(P_1) \subset P_2$
2. The functor F is said to reflect P, if for any $g \in P_2$ the solutions to the lifting problems defined by g are diagrams of morphisms in P_1.
3. The functor F creates P, if F reflects P and any lifting problem diagram defined by a morphism $g \in P_2$ is solved by a diagram of some morphism $f \in P_1$.

[4] A pair of morphisms is said to parallel, if they are elements of the same hom-set.

4. We choose to say that the functor F creates P strictly, if all lifting problem defined by any morphism in P_2 has a unique solution and furthermore this solution diagram is defined by a morphism in P_1.

We may leave the pair P to be implicitly defined. We may be interested in knowing when a functor preserves epimorphisms. In that case P_1 will be implicitly defined to be the epimorphisms of the domain category and P_2 the epimorphisms of the codomain category.

It makes sense to talk about functor preserving properties of objects, since objects in a category can be identified with the corresponding identities.

Theorem 2.26. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. Then the following hold:*

1. *The functor F preserves retractions, sections and isomorphisms.*

2. *The functor F need not preserve monomorphisms or epimorphisms.*

3. *The functor F doesn't necessarily reflect isomorphisms.*

Proof.

1. By duality and, since isomorphisms are exactly those morphisms that are sections and retractions, it suffices to show that retractions are preserved. Assume that r is a retraction in $\mathbf{C}$. Thus there exists s in $\mathbf{C}$ where $rs = id$. Hence

$$F(r)F(s) = F(id) = id.$$

Therefore $F(r)$ is a retraction.

2. Consider the set function $\{1,2\} \xrightarrow{f} \{1,2\}, f \equiv 1$ of the category of sets. The diagram $\overline{f}$ defined by f does not preserve monomorphisms nor epimorphisms.

3. Any diagram defined by an identity morphism does not reflect isomorphisms.

□

Definition 2.27. We say that a functor $F : \mathbf{C} \to \mathbf{D}$ is dense, if for every object d in $\mathbf{D}$ there exists an object c in $\mathbf{C}$ such that $Fc \cong d$.

Theorem 2.28. *Let $\boldsymbol{C} \xrightarrow{F} \boldsymbol{D} \xrightarrow{G} \boldsymbol{E}$ be functors. Then the following hold:*

1. *If GF is faithful, then so is F.*

2. *If GF is dense, then G is dense too.*

3. *If GF is full and F is dense, then G is full.*

Proof.

1. Assume GF is faithful. Let a and b be objects in $\mathbf{C}$ and let $f, g : a \to b$ be morphisms. Assume $F(g) = F(f)$. Now $GFf = GFg$ and hence $f = g$.

2. Assume that GF is dense. Let d be an object in $\mathbf{D}$. Now there exists an object in $\mathbf{C}$ where $GF(c) \cong d$. Hence $G(F(c)) \cong d$. Thus G is dense.

3. Let GF be full, F dense and d, d' be objects in $\mathbf{D}$. Assume that $h : G(d) \to G(d')$ is a morphism in $\mathbf{E}$. We need to show that there exists a morphism $g : d \to d'$ that G maps to h. There exist objects c and c' in $\mathbf{C}$ and isomorphisms $k : F(c) \cong d$ and $l : F(c') \cong d'$. Define $h' : GF(c) \to GF(c')$ to be the unique morphism that makes the diagram

$$\begin{tikzcd} GF(c) \arrow[r, "h'"] \arrow[d, "Gk"] & GF(c') \arrow[d, "Gl"] \\ Gd \arrow[r, "h"'] & Gd' \end{tikzcd} \tag{2.1}$$

commute. There exists $f : c \to c'$ where $GFf = h'$. Henceforth define $g : d \to d'$ to be the unique morphism that makes the diagram

$$\begin{tikzcd} F(c) \arrow[r, "Ff"] \arrow[d, dashed, "k"] & F(c') \arrow[d, dashed, "l"] \\ d \arrow[r, "g"'] & d' \end{tikzcd}$$

commute. Replacing h by Gg in the diagram (2.1), we see that $G(g)$ makes the diagram commute. By uniqueness $h = G(g)$.

□

Theorem 2.29. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. The the following hold:*

1. *If F is faithful, then F reflects epimorphisms and monomorphisms.*
2. *If F is fully faithful, then F strictly creates sections, retractions and isomorphisms. Additionally, a fully faithful F reflects initial and terminal objects.*

Proof.

1. Assume that F is faithful. It suffices to show, by duality, that F reflects monomorphisms. Let $f : a \to b$ be a morphism in $\mathbf{C}$ and assume that Ff is a monomorphism. Let $x, x' : c \to a$ be morphisms where $fx = fx'$. Thus the diagrammatic lifting problem

$$\begin{tikzcd} Fc \arrow[r, bend left, "Fx"] \arrow[r, bend right, "Fx'"'] & Fa \arrow[r, "Ff"] & Fb \end{tikzcd}$$

commutes. By faithfulness the unique solution commutes (Theorem 2.18). Thus $x = x'$.

2. Assume that F is fully faithful. The isomorphism case follows from the other two cases. By duality, it suffices to show that F strictly creates sections. Let $s : Fa \to Fb$ be a section and let $r : Fb \to Fa$ be a left inverse of s. By fully faithfulness, the commutative lifting problem

$$\begin{tikzcd} & Fa \\ Fa \arrow[ur, "id_{Fa}"] \arrow[r, "s"'] & Fb \arrow[u, dashed, "r"'] \end{tikzcd}$$

has a unique solution that commutes. Thus F strictly creates sections.

Lastly, assume that Fc is an initial object, where c is an object in $\mathbf{C}$. Now $\mathrm{Hom}(c, x) \cong \mathrm{Hom}(Fa, Fx)$ for all objects x in $\mathbf{C}$. Hence c is initial. Similarly for terminal objects. Hence F reflects terminal and initial objects.

□

2.5.2 Topological category

There is a special class of epimorphisms called identifications. The dual notion of an identification is an embedding. These notions of embedding and identification are closely related to the idea of inductance of structure. We briefly introduce the concept of a topological functors, which are very special faithful functors that reflect a lot of structure from the codomain category. In this section we follow loosely the book "Abstract and Concrete Categories. The Joy of Cats" by Jiří Adámek, Horst Herrlich and George E. Strecker [1].

Definition 2.30 (Concrete category)**.** Let $\mathbf{C}$ be a category and let $F : \mathbf{C} \to \mathbf{D}$ be a faithful functor. Then we call the pair $(\mathbf{C}, F)$ a concrete category over $\mathbf{D}$. A concrete category over the category of sets is called a concrete category.

Example 2.31.

1. Usually forgeful functors are faithful and hence define a cocrete category over the codomain category. As an example consider the category $\mathbf{Model}_L^T$ of L, T-models, where L is an alphabet and T an L-theory, and the forgetful functor $\mathbf{Model}_L^T \to \mathbf{Set}$. Similarly the forgetful functor from the category of topological spaces to the category of sets defines a concrete structure.

2. Every proset category is a concrete category over the terminal category $\mathbf{1}$.

3. The category **Set** has two concrete structures defined by the identity functor and the covariant power set functor.

Definition 2.32 (Concrete functor)**.** Let $(\mathbf{C}, F)$ and $(\mathbf{D}, G)$ be concrete categories over $\mathbf{I}$. A functor $L : \mathbf{C} \to \mathbf{D}$ that lifts F along G is called concrete functor.

We see that every concrete functor is faithful and uniquely specified by mapping of objects by Theorem 2.18.

Definition 2.33. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. Let I be a moderate set and let x, y be objects in $\mathbf{D}$ and let x_i, y_i be objects in $\mathbf{D}$ and $\mathbf{C}$, respectively, for $i \in I$. Let $f_i : x \to F(y_i)$ and $g_i : F(x_i) \to y$ be morphisms in $\mathbf{D}$ for $i \in I$. Denote $f = (f_i : x \to F(y_i))_{i\in I}$ and $g = (g_i : F(x_i) \to y)_{i\in I}$. The moderate collection of morphisms f associated with x and $(y_i)_{i\in I}$ is called an F, I-source. Similarly the collection of morphisms g associated with y and $(x_i)_{i\in I}$ is called an F, I-sink.

Definition 2.34 (Induced structure via morphisms)**.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let I be a moderate set. Let $f = (f_i : x \to Fy_i)_{i\in I}$ be an F, I-source. Define a category $\mathbf{A}$, called the structure category with respect to F, I and f, as follows:

- The objects are tuples (a, α, p) where a is an object of $\mathbf{C}$, $\alpha = (\alpha_i : a \to y_i)_{i\in I}$ is a collection of morphisms in $\mathbf{C}$ and $p : Fa \to x$ is a morphisms such that the diagram

$$\begin{tikzcd} Fa \arrow[r, "p"] \arrow[dr, "F\alpha_i"'] & x \arrow[d, dashed, "f_i"] \\ & Fy_i \end{tikzcd}$$

commutes for all $i \in I$.

- A morphism $\theta : (a, \alpha, p) \to (b, \beta, q)$ is a morphism $\theta : a \to b$ in $\mathbf{C}$ where the diagrams

$$\begin{tikzcd} & Fb \arrow[d, dashed, "q"] \\ Fa \arrow[ur, "F\theta"] \arrow[r, "p"'] & x \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[d, dashed, "\beta_i"] \\ a \arrow[ur, "\theta"] \arrow[r, "\alpha_i"'] & y_i \end{tikzcd}$$

commute for every $i \in I$.

- The composition is defined in the usual way.

Let c be an object in $\mathbf{C}$, where $Fc = x$, and let $\alpha = (\alpha_i : c \to y_i)_{i\in I}$ be a collection of morphisms such that $F\alpha_i = f_i$ for all $i \in I$. We say that the F-source (x, f) F-induces the pair (c, α), if (c, α, id_x) is terminal in $\mathbf{A}$. Furthermore, we say that α F-induces the structure on c.

The previous definition was done with arbitrary categories and functors. Thus the construction dualizes. Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let I be a moderate set. Assume that x_i and y are objects in $\mathbf{C}$ and $\mathbf{D}$, respectively, for $i \in I$. Assume that $f = (f_i : Fx_i \to y)$ is an F, I-sink. We say that the collection of morphisms f F-coinduces a $\mathbf{C}$-structure on y, if $f^{op} := (f^{op} : y \to Fx_i)_{i\in I}$ F^{op}-induces a $\mathbf{C}^{op}$-structure.

Definition 2.35. A functor $F : \mathbf{C} \to \mathbf{D}$ is called topological and the pair $(\mathbf{C}, F)$ a topological category over $\mathbf{D}$, if every F-source F-induces a $\mathbf{C}$-structure. A topological category is just a topological category over the category of sets.

Notice that every proset category with arbitrary infimums is a topological category over the terminal category $\mathbf{1}$.

Theorem 2.36. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a faithful functor, and let $f = (f_i : x \to F(y_i))_{i \in I}$ be an F, I-source, where I is a moderate set. Let $\boldsymbol{A}$ be the structure category with respect to F,I and f. Then the source f induces a $\boldsymbol{C}$-structure on x with respect to the functor F, if and only if the following holds:*

There exists an object c in $\boldsymbol{C}$ such that $F(c) = x$. Assume that $h : F(z) \to x$ and $h_i : F(z) \to F(y_i)$ are arbitrary morphisms where the diagram

$$\begin{tikzcd} F(z) \arrow[r, "h"] \arrow[dr, "h_i"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \end{tikzcd}$$

commutes for every $i \in I$. Then F lifts $F(z) \xrightarrow{f} F(c)$, if and only if F lifts the diagram $F(z) \xrightarrow{h_i} F(y_i)$ for every $i \in I$. Furthermore, the structure induced on x is c.

Proof. Assume that f induces a $\mathbf{C}$-structure on x. Hence there exists an object c in $\mathbf{C}$, where $F(c) = x$ and $\alpha = (\alpha_i : c \to y_i)_{i \in I}$, where $F\alpha_i = f_i$ and (c, α, id_x) is terminal in the structure category. Assume that the diagram

$$\begin{tikzcd} F(z) \arrow[r, "h"] \arrow[dr, "h_i"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \end{tikzcd}$$

in $\mathbf{D}$ commutes for all $i \in I$. It is clear that, if F lifts $F(z) \xrightarrow{h} F(c)$, then F lifts $F(x) \xrightarrow{h_i} F(y_i)$ for all $i \in I$. Assume the converse. Hence there exist $\beta_i : z \to y_i$, where $F(\beta_i) = h_i$. Set $\beta := (\beta_i)_{i \in I}$. Thus (z, β, h) is an object in the structure category. By the terminality of (c, α, id_x) it follows that there exists $\theta : z \to c$ where $id_x F(\theta) = h$. Hence the diagram $F(z) \xrightarrow{h} F(c)$ is lifted by F.

Assume that there exists c in $\mathbf{C}$ where $F(c) = x$. Assume also that when ever a collection of diagrams

$$\begin{tikzcd} F(z) \arrow[r, "h"] \arrow[dr, "h_i"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \end{tikzcd}$$

in $\mathbf{D}$ commutes for $i \in I$, then the diagram for h is lifted, if and only if the diagrams for $h_i, i \in I$ are lifted.

Now there is a commutative diagram

$$\begin{tikzcd} F(c) \arrow[r, "id_{F(c)}"] \arrow[dr, "f_i"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \end{tikzcd}$$

and the diagram for the identity is lifted. Thus there are morphisms $\alpha = (\alpha_i : c \to y_i)_{i \in I}$ where $F(\alpha_i) = f_i$. Thus (c, α, id_x) is an object in the structure category. We need to show that it is terminal.

Let (z, β, h) be an object in the structure category. We need to show that there exists a unique morphism $\theta : (z, \beta, h) \to (c, \alpha, id_x)$. The uniqueness of θ follows from the faithfulness of F. Thus the diagram

$$\begin{tikzcd} F(z) \arrow[r, "h"] \arrow[dr, "h_i"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \end{tikzcd}$$

commutes where $h_i = F(\beta_i)$. By assumption there is a lift for the diagram of h and so there exists a morphism $\theta : z \to c$ where $F(\theta) = h$. Hence $id_x F(\theta) = h$ and so the first condition holds for θ to be a morphism in the structure category. Now

$$\begin{aligned} F(\alpha_i\theta) &= F(\alpha_i)F(\theta) \\ &= f_i h \\ &= h_i \\ &= F(\beta_i). \end{aligned}$$

Since F is faithful, $\alpha_i\theta = \beta_i$. Hence $\theta : (z, \beta, h) \to (c, \alpha, id_x)$. Lastly, the **C**-structure induced on x is c. □

The previous theorem dualizes, since the faithfulness of F guarantees the faithfulness of F^{op}.

Example 2.37.

1. Let $F : \mathbf{C} \to \mathbf{D}$ be a faithful functor. Then every section $s : x \to y$ in $\mathbf{C}$ induces the structure on x with respect to the functor F. Assume that $g : a \to y$ is a morphisms in $\mathbf{C}$ and a diagram
$$\begin{tikzcd} Fa \arrow[r, "f"] \arrow[dr, "Fg"'] & Fx \arrow[d, dashed, "Fs"] \\ & Fy \end{tikzcd}$$
in $\mathbf{D}$ commutes. It suffices to see that the diagram of the morphism f is lifted. Denote a left inverse of s by r. Define $\theta := rg$. Now
$$F\theta = FrFg = FrFsf = f.$$
Thus the diagram of f is lifted. Hence s inuduces the structure on x.

2. The category **Top** of topological spaces is a topological category. Let X be a small set, let I be a moderate set, let Y_i be a topological space and let $f_i : X \to Y_i$ be a function for $i \in I$. The functions f_i induce a topological structure on X by defining the smallest topology τ on X where the functions f_i are continuous. The set τ consists of arbitrary unions of finite intersections of sets of the form $f_i^{-1}(V)$ where $V \subset Y$ are open, $i \in I$.

   Let Z be a topological space and let $g : Z \to X$ be a function. To check the continuity of $g : Z \to X$, it suffices to check that $g^{-1}(f_i^{-1}(V))$ is open for every open $V \subset Y$ and $i \in I$. Thus the continuity of $f_i \circ g, i \in I$, implies the continuity of g and the converse also holds for all functions $g : Z \to X$. This means exactly the same that the functions f_i induce **Top**-structure on X with respect to the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. Thus the forgetful functor U is a topological functor.

3. The category of measurable spaces is topological. This is seen by slightly modifying the argument about topological spaces.

4. The category of topological groups is topological over the category of groups.

Theorem 2.38. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories and let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a topological functor. Then the following claims hold:*

1. *The functor F is faithful.*

2. *The functor $F^{op} : \boldsymbol{C}^{op} \to \boldsymbol{D}^{op}$ is a topological functor. (The Topological Duality Theorem)*

Proof.

1. Let $f, g : a \to b$ be morphisms in $\mathbf{C}$ and assume that $Ff = Fg$. Consider the $F, \mathrm{Mor}(\mathbf{C})$-source $(Fa \xrightarrow{Ff} Fb)_{i \in \mathrm{Mor}(\mathbf{C})}$ and the corresponding induced structure $(a' \xrightarrow{h_i} b)_{i \in \mathrm{Mor}(\mathbf{C})}$. Furthermore, consider the collection of morphisms $(t_i : a \to b)_{i \in \mathrm{Mor}(\mathbf{C})}$, where
$$t_i = \begin{cases} f, \text{ if } h_i \circ i = g \\ g, \text{ else} \end{cases}$$
for $i \in \mathrm{Mor}(\mathbf{C})$. The object $(a, (t_i)_{i \in \mathrm{Mor}(\mathbf{C})}, id_{Fa})$ is an object in the structure category and thus there exists a unique morphism $\theta : (a, (t_i)_i, id_{Fa}) \to (a', (h_i)_i, id_{Fa})$. Thus $h_i\theta = t_i$ for every $i \in \mathrm{Mor}(\mathbf{C})$. Therefore $h_\theta\theta = t_\theta$. If $f \neq g$, then by the definition of t_θ, the morphism t_θ cannot be either f nor g, which yields a contradiction. Thus $f = g$.

2. Let I be a moderate set. Assume that $f = (Fx_i \xrightarrow{f_i} y)_{i\in I}$ is an F-sink. We need to show that f F-coinduces a structure on y. Consider the moderate collection $h = (y \xrightarrow{h_i} Fb_i)_{i\in J}$ of all the morphisms $h' : y \to Fb$ in $\mathbf{D}$, where b is an object in $\mathbf{C}$ and the diagram $Fx_i \xrightarrow{hf_i} Fb$ is lifted by F for each $i \in I$. Consider the induced structure $(a \xrightarrow{\alpha_i} b_i)_{i\in J}$ defined by h. Assume that the diagram

$$\begin{tikzcd} Fx_i \arrow[r, "t_i"] \arrow[d, dashed, "f_i"'] & Fz \\ Fa \arrow[ur, "t"'] & \end{tikzcd}$$

commutes in $\mathbf{D}$ for every $i \in K$, where K is a moderate set. Consider the commutative diagram

$$\begin{tikzcd} Fx_i \arrow[r, "f_i"] \arrow[dr, "h_jf_i"'] & Fa \arrow[d, "h_j"] \\ & Fb_j \end{tikzcd}$$

for $i \in I$ and $j \in J$. By definition the diagram of h_jf_i is lifted for every $j \in J$ and thus the diagram of f_i is lifted. If the diagram of t is lifted, so is t_i for every $i \in I$. Assume that the diagram of t_i is lifted for every $i \in I$. Then there exists indices $j_i \in J$, where $t_i = h_{j_i}$. Now $F\alpha_{j_i} = t_i$ for every $i \in I$. Thus F^{op} is a topological functor.

□

By The topological duality Theorem 2.38(2) shows the if we are always capable of inducing a structure, then we are always capable of coinducing a structure. Hence we attain that the functions $g_i : X_i \to Y, i \in I$ coinduce a topological structure on the set Y, where X_i is a topological space for every $i \in I$. The coinduced topology on Y is the biggest topology on Y, where the functions g_i are continuous.

From The Topological Duality Theorem we attain that if a proset category $\mathbf{P}$ is topological over the terminal category $\mathbf{1}$, then the any subset of objects of $\mathbf{P}$ has a supremum.

Theorem 2.39. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a faithful functor. Let $p : c \to d$ be a morphism in $\boldsymbol{C}$ and assume that p coinduces (induces) the structure on d (c) and Fp is an isomorphism. Then p is an isomorphism.*

Proof. Since F is faithful and Fp is an epimorphism, it follows that p is an epimorphism. Consider the commutative diagram

$$\begin{tikzcd} Fc \arrow[r, "Fid_c"] \arrow[d, dashed, "Fp"'] & Fc \\ Fd \arrow[ur, "(Fp)^{-1}"'] & \end{tikzcd}$$

and since p coinduces the structure, there exists $g : d \to c$, where $Fg = (Fp)^{-1}$. Thus $gp = id_c$, by the faithfulness of F, and so p is an epic section. Hence p is an isomorphism. □

Even though bijective continuous map might not be a homeomorphism, a bijective continuous map that coinduces the structure on the image is a homeomorphism.

Theorem 2.40 (Transitivity of inductance)**.** *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a faithful functor and let I and J be moderate sets. Fix a collection of morphisms $x \xrightarrow{f_i} F(y_i) \xrightarrow{g_{i,j}} F(z_{i,j}), (i, j) \in I\times J$. Assume that the morphisms $g_{i,j} : F(y_i) \to z_{i,j}, j \in J$, induce the structure y_i for all $i \in I$. Then $(g_{i,j}f_i)_{i\in I\times J}$ induces a structure, if and only if $(f_i)_{i\in I}$ induces a structure. Additionally, the structures are the same up to a unique isomorphism.*

Proof. Let the following be a commutative diagram in $\mathbf{D}$ for every $(i,j) \in I \times J$:

$$\begin{tikzcd} Fa \arrow[r, "h"] \arrow[dr, "h_i"] \arrow[ddr, "h_{i,j}"'] & F(c) \arrow[d, dashed, "f_i"] \\ & F(y_i) \arrow[d, dashed, "g_{i,j}"] \\ & F(z_{i,j}) \end{tikzcd}$$

Assume first that the collection $(f_i)_i$ induces a structure c on x. The top of the diagram being lifted is equivalent with the middle being lifted for all $i \in I$ which is again equivalent with the bottom being lifted for $(i,j) \in I \times J$. Hence $(g_{i,j}f_i)_{(i,j)\in I\times J}$ induces the structure c on x.

Assume that $(g_{i,j}f_i)_{i,j}$ induces a structure c on x. The top being lifted is equivalent with the bottom being lifted for $(i,j) \in I \times J$ which is equivalent with the middle being lifted for $i \in I$. Hence $(f_i)_{i\in I}$ defines the structure c on x. ☐

2.5.3 Embedding and identification

Definition 2.41. Let $(\mathbf{C}, F)$ be a concrete category and let $i : c \to d$ be a morphism in $\mathbf{C}$. We call the morphism i an embedding, if the function Fi is monic and i induces the structure on c. Dually, we call a morphism $f : c \to d$ in $\mathbf{C}$ an identification, if Ff is epic and f coinduces the structure on d with respect to the functor F.

Corollary 2.42. *Assume that (C, F) is a concrete category and the diagram*

$$\begin{tikzcd} b \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & c \arrow[d, dashed, "i"] \\ & d \end{tikzcd}$$

commutes in $\mathbf{C}$. Assume that i is an embedding. Then f is an embedding, if and only if $\tilde{f}$ is.

Proof. The claim follows directly from the transitivity of inductance. ☐

The corresponding case holds for identifications by duality.

Example 2.43.

1. Let $(\mathbf{C}, F)$ be a concrete category and let $f : x \to y$ be a morphism in $\mathbf{C}$. Assume that the quotient map $q : Fx \to Fx/\sim$ coinduces the $\mathbf{C}$-structure on $Fx/\sim$, where $\sim$ is the set theoretical kernel of Ff. Denote the coinduced morphism by $p : x \to x/\sim$. Notice that there is a commutative diagram

$$\begin{tikzcd} Fx \arrow[r, "Ff"] \arrow[d, dashed, "Fp"] & Fy \\ Fx/\sim \arrow[ur] & \end{tikzcd}$$

   Since p is an identification, there is a unique morphism $t : x/\sim \to y$ where $tp = f$.

2. Let L be an alphabet and let T be an L-theory. Consider the concrete category of L,T-models, $\mathbf{Model}_L^T$, with the canonical forgetful functor $(X,T) \mapsto X$. Then every globally full surjective model morphism in $\mathbf{Model}_L^T$ is an identification. The converse holds, if T is a positive theory. To this end, fix a surjective model morphism $p : \mathcal{M} \to \mathcal{N}$, where M and N denote the corresponding universes. Let (Z, C) an object in $\mathbf{Model}_L^T$. Consider the commutative diagram in $\mathbf{Set}$

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, dashed, "p"] & Z \\ N \arrow[ur, "\tilde{g}"'] & \end{tikzcd}$$

If p is globally full and surjective, then by Lemma 1.44(1) it follows that g is a model morphism, if and only if $\tilde{g}$ is. Hence p is an identification.

For the converse, assume that $p : \mathcal{M} \to \mathcal{N}$ is an identification and T is a positive theory. Since T is a positive theory, by The Fundamental Theorem of Model Morphisms, the quotient $q : \mathcal{M} \to \mathcal{M}/\ker(p)$ is a model morphism in $\mathbf{Model}_L^T$. Consider the commutative diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "p"] \arrow[d, dashed, "q"] & N \\
\mathcal{M}/\ker(p) \arrow[ur, "\tilde{p}"'] &
\end{tikzcd}$$

Because p and q are identifications, so is $\tilde{p}$. Since $\tilde{p}$ is bijective identification, $\tilde{p}$ is an isomorphism. Since q is globally full, so is p.

3. Let L be an alphabet and let T be an L-theory. Let $\mathcal{M}$ and $\mathcal{N}$ be L-models with universes M and N, respectively. Assume that $f : \mathcal{M} \to \mathcal{N}$ is an L-model morphism. Then f is an embedding, if it is globally full. The converse holds, if the theory T is positive.

   Assume that $\mathcal{X}$ is an L-model with universe X and $g : X \to M$ is a function where $fg : \mathcal{X} \to \mathcal{N}$ is an L-model morphism. It suffices to show that g is a model morphism $\mathcal{X} \to \mathcal{M}$. By the injectivity of f, $g(c^{\mathcal{X}}) = c^{\mathcal{M}}$ for all constant symbols c of L. If α is a function symbol, then

$$fg\alpha^{\mathcal{X}} = \alpha^{\mathcal{N}}(fg)_* = \alpha^{\mathcal{N}} f_* g_* = f\alpha^{\mathcal{M}} g_*$$

   and so, by the injectivity of f, $g\alpha^{\mathcal{X}} = \alpha^{\mathcal{M}} g_*$. Lastly, fix a relation symbol R of L. Then

$$f_* g_*[R^{\mathcal{X}}] = (fg)_*[R^{\mathcal{X}}] \subset R^{\mathcal{N}} \cap \mathrm{im}(f_*)$$

   and by the injectivity and global fullness of f_*, it follows that $g_*[R^{\mathcal{X}}] \subset R^{\mathcal{M}}$. Hence g is a model morphism. This shows that f is an embedding with respect to the canonical forgetful functor $\mathbf{Model}_L^T \to \mathbf{Set}$.

   For the converse assume that T is a positive theory and assume that f is an embedding. Since f is injective, f is an isomorphism to its image $\mathrm{im}(f)$. Henceforth consider the commutative diagram:

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[r, "f|^{-1}"] \arrow[dr, hook] & \mathcal{M} \arrow[d, "f"] \\
& \mathcal{N}
\end{tikzcd}$$

   Because $f\ |^{-1}$ and f are embeddings, so is the inclusion $\mathrm{im}(f) \hookrightarrow \mathcal{N}$. Consider the full submodel $\mathcal{X}$ of $\mathcal{N}$ defined by the set theoretic image $\mathrm{im}(f)$ of f. Now $\mathrm{im}(f)$ is a submodel of $\mathcal{X}$ and the inclusion $\mathrm{im}(f) \hookrightarrow \mathcal{X}$ is surjective. Hence $\mathcal{X} \vDash T$, because the theory T is positive. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{im}(f) \arrow[r, hook, "j"] \arrow[dr, hook, "i"'] & \mathcal{X} \arrow[d, hook, "k"] \\
& \mathcal{N}
\end{tikzcd}$$

   and, since k and i are embeddings, so is j. Since j is bijective embedding, it is an isomorphism. Thus f is globally full.

Let L and T be the vocabulary and theory of categories. Since functors between small categories can be identified as model morphisms in $\mathbf{Model}_L^T$, we see that a fully faithful functor that is injective on objects is an embedding with respect to the canonical forgetful functor $\mathbf{Cat} \to \mathbf{Set}$.

2.6 Construction of objects

In this section we are going find structure in an object and construct operations, such as product, that have a very algebraic flavour to them. Categorical product unifies the ideas of minimum, intersection, union and the greatest common divisor.

2.6.1 Subobjects

In category theory, one categorizes concepts that repeat in the fields of mathematics. The idea of a subobject repeats all over mathematics, but it is quite hard to give a satisfying categorical definition for it. If X is a set, then its subobjects are subsets $A \subset X$ and those can be identified with their inclusions $A \hookrightarrow X$. In an abstract category we are not able to differentiate between isomorphic objects, so we aren't capable of choosing some injection over an other. Equivalence relations helps us with this conundrum. We may define such an equivalence relation $\sim$ of injections that in every equivalence class there lives one and only one inclusion. We want to define an equivalence relation $\sim$ of injections f and g with codomain X, where $f \sim g$, if and only if $\mathrm{im} f = \mathrm{im} g$. We are able to define such an equivalence relation is a general setting.

Definition 2.44. Let $\mathbf{C}$ be a category. We define a preorder structure on the set of monomorphisms Mon_a with codomain a, by setting $(x \xrightarrow{i} a) \leq (y \xrightarrow{j} a)$, if and only if there exists a morphism $k : x \to y$ in $\mathbf{C}$ that makes the diagram

$$\begin{tikzcd} & y \arrow[d, "j"] \\ x \arrow[ur, "\exists k"] \arrow[r, "i"'] & a \end{tikzcd} \tag{2.2}$$

commute. Hence we get a poset $\mathrm{Mon}_a/\sim$, denoted $\mathrm{Sub}(a)$, where we equivalete monics $i, j \in \mathrm{Mon}_a$, where $i \leq j$ and $j \leq i$. The elements of the poset $\mathrm{Sub}(a)$ are called the subobjects of a.

If such a k exists, then by monicness of j, k must be monic. Additionally, if the monomorphisms i and j are equivalent, the morphism $k : x \to y$ will be an isomorphism.

Example 2.45.

1. If X is a set, then there is a bijection with the set of subobjects of X, $\mathrm{Sub}(X)$, and the set of subsets of X, $\mathcal{P}(X)$, by mapping a subset $A \subset X$ to the equivalence class of the inclusion $i : A \hookrightarrow X$. Bijectivity is seen as follows: If $f : A \to X$ and $g : B \to X$ are injections, then

$$f \leq g, \text{ if and only if } \mathrm{im} f \subset \mathrm{im} g. \tag{2.3}$$

   Consider the equivalent inclusions $A \hookrightarrow X$ and $B \hookrightarrow X$. Since they are equivalent, they share the same image. Hence $A = B$. Surjectivity is also seen, since given any injection $A \xrightarrow{f} X$, we have it to be equivalent with the inclusion $\mathrm{im} f \hookrightarrow X$, since the images match.

2. If X is a topological space, then every subspace of X corresponds to a subset A of X, with the topology induced by the inclusion $A \hookrightarrow X$. The previous characterization (2.3) of equivalent monomorphisms by the images only applies from left to right, since there exists a space X where there exists injections i, j to X and the images match, but i and j are not equivalent.

   For example, choose the space X to be the set $2 = \{0, 1\}$ with the topology $\{\emptyset, X, \{0\}\}$. The space X is called the Sierpinski space. Let A be the space $\{0, 1\}$ with the discrete topology. The identity on set 2 is a continuous monomorphism $A \to X$, since any map from a discrete space is continuous. Additionally, the identity on X is a continuous map. The injections $X \to X$ and $A \to X$ cannot be equivalent, since then X and A would be homeomorphic.

   Furthermore, there is an injection $\mathcal{P}(X) \to \mathrm{Sub}(X)$ by setting $A \mapsto [A \hookrightarrow X]$. For the injectivity it is enough to prove the direction from left to right of the equivalence in (2.3). This shows that the cardinalities of $\mathcal{P}(X)$ and $\mathrm{Sub}(X)$ do not match in our previous example. Hence no bijection can exist between the two posets.

3. Let L be an alphabet with a theory T. Let $\mathcal{M}$ be an L, T-model. Then the association

$$\mathcal{A} \overset{F}{\longmapsto} [\mathcal{A} \hookrightarrow \mathcal{M}],$$

   for every submodel $\mathcal{A} \vDash \mathcal{T}$ of $\mathcal{M}$, defines an injection $\{\mathcal{A} \mid \mathcal{A} \text{ is a submodel of } \mathcal{M}\} \to \mathrm{Sub}(\mathcal{M})$. The function F is bijective, if all monics are injective. The function F is injective, since if two inclusions i and j are equivalent, then the connecting isomorphism $k : i \cong j$ must be the identity.

Assume that all monics of $\mathbf{Model_L^T}$ are injections. We show that F is surjective. Let $f : \mathcal{A} \to \mathcal{M}$ be a monic L, T-model morphism. Thus f is injective. Now the corestriction $f| : \mathcal{A} \to \mathrm{im}(f)$ defines a globally full bijection and hence an isomorphism. Thus $\mathrm{im}(f) \vDash T$. Now the inclusion $\mathrm{im}(f) \hookrightarrow \mathcal{M}$ is equivalent with f via $f|$. Therefore F is surjective.

This shows that in the categories $\mathbf{Mon}, \mathbf{Grp}$ and $\mathbf{R}$-$\mathbf{Mod}$ the subobjects correspond to submonoids, subgroups and submodules, respectively.

In the previous examples, we notice that the image of a function plays an important role when studying the subobjects. This motivates us to give a categorical definition of an image of a morphism. We define the image of a morphism f to be the smallest subobject of the codomain of f through which f factors:

Definition 2.46. Let $f : c \to d$ be a morphism in a category $\mathbf{C}$. Let $i : d' \to d$ be a monomorphism. We say that i is the (categorical) image of f, if there exists $f' : c \to d'$, where the diagram

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "f'"'] & & d \\ & d' \arrow[ur, "i"'] & \end{tikzcd}$$

commutes and i is initial of such factorizations. Specifically, we require that given any other factorization $f = jf''$ along a monomorphism $j : d'' \to d$, then there exists a unique morphism $k : d' \to d''$ where the diagram

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "f'"'] \arrow[ddr, bend right, "\forall f''"'] & & d \\ & d' \arrow[ur, "i"'] \arrow[d, dotted, "\exists! k"] & \\ & \forall d'' \arrow[uur, bend right, "\forall j"'] & \end{tikzcd}$$

commutes.

Since the morphism k connects two monomorphisms, k is unique and a monomorphism. The left triangle commutes automatically without needing to assume it, since

$$jkf' = if' = f = jf''$$

and thus $kf' = f''$ by the monicness of j. Hence it suffices to say that $i \leq j$. In a factorization $f = if'$, the morphism f' is called a codomain restriction of f and the morphism f' is unique by the monicness of i. Colloquially we may say that f' is a corestriction of f, but this is not a dual concept to the restriction of f, which is fi for some subobject i of the domain of f.

If a morphism $f : c \to d$ has the identity id_d as its image, then we say that f has a full image. Since we may identify the object d with the identity $id_d : d \to d$, we say that the categorical image of f is d, if f has a full image. A morphism that has a full image and is an epimorphism is called an extremal epimorphism.

If f is both an extremal epimorphism and a monomorphism, then f is an isomorphism: Assume that f is a monic extremal epimorphism. Now $f \leq id_d$ and since $f = fid_c$, we have $id_d \leq f$ and thus f is an isomorphism.

Example 2.47. Let $\mathbf{Model}_L^T$ be a positive category where all monics are injective. Let $f : \mathcal{M} \to \mathcal{N}$ be a morphism in $\mathbf{Model}_L^T$. Since the image of f is a submodel of $\mathcal{N}$ and f becomes a surjection on $\mathrm{im}(f)$, it follows that $\mathrm{im}(f)$ also satisfies the theory T. We claim that the inclusion $\mathrm{im}(f) \hookrightarrow \mathcal{N}$ is the categorical image of f.

We define $f' : \mathcal{M} \to \mathrm{im}(f)$, $m \mapsto f(m)$. The function f' is an identification in $\mathbf{Model}_L^T$. Now $f = if'$ where i is monic. Assume that there is an other representation $f = jf''$, where $f'' : \mathcal{M} \to \mathcal{X}$ and j is monic. Since that $f = jf''$ and j is an injection, $\ker(f') = \ker(f'')$. Because $\ker(f') = \ker(f'')$ and f' is a globally full surjection, there exists a unique L-model morphism

$k : \mathrm{im} f \to X$, where the left side of the diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "f"] \arrow[dr, "f'"'] \arrow[ddr, bend right, "f''"'] & & \mathcal{N} \\
& \mathrm{im}(f) \arrow[ur, "i"'] \arrow[d, dashed, "k"] & \\
& X \arrow[uur, bend right, "j"'] &
\end{tikzcd}$$

commutes. By the epicness of f' the right side of the diagram commutes also, because

$$jkf' = jf'' = f = if'$$

and so $jk = i$.

Now we see that the categorical image is defined for all R-linear maps, group homomorphisms, monoid homomorphisms and set functions.

2.6.2 Quotient objects

Since the definition of a subobject was done in an arbitrary category, by fixing a category $\mathbf{C}$, we can look at the definition of a subobject in $\mathbf{C}^{op}$ and pull it back to $\mathbf{C}$. Hence we get the concept of a quotient object. The study of subobjects gave us a method of finding internal structure on an object. With groups, for example, it is quite intuitive to look for more groups in a given one, but one may ask if there are more structure than just subgroups. The other groups hiding inside a group, up to a suitable equivalation of elements, are the quotient groups. Every subgroup of the integers $\mathbb{Z}$ looks either like the trivial group or $\mathbb{Z}$ itself, but the quotient groups are more interesting. Every cyclic group is isomorphic to exactly one quotient group of $\mathbb{Z}$. Intuitively a quotient group of c, and more generally a quotient set of c, is a new object d obtained from collapsing of points of c so that d inherits the object structure from c. Hence partitions, equivalently equivalence relations, are closely related to quotient objects.

Definition 2.48. Let $(\mathbf{C}, F)$ be a concrete category. Let c be an object of $\mathbf{C}$ and assume that f is a morphism in $\mathbf{C}$ with domain c. We call the set theoretical kernel of Ff an F-congruence on c. The class of all F-congruences on c is denoted $\mathrm{Cong}_F(c)$. If the functor F clear from the context, we may not mention it.

Fix an alphabet L and a positive L-theory T. It is worth noting that the congruences of $\mathbf{Model}_L^T$ with respect to the canonical forgetful functor coincide with the definition of L-model congruence by Theorems 1.44(2) and 1.46.

Definition 2.49. Let a be an object in a category $\mathbf{C}$. We call the subobjects of a in the category $\mathbf{C}^{op}$ the quotient objects of a. To be precise we define a preorder structure on epimorphisms with the domain a as follows: Let $s : a \to x$ and $t : a \to y$ be epimorphisms. We say that $s \geq t$, if there exists a morphism $k : x \to y$ where the diagram

$$\begin{tikzcd}
a \arrow[r, "s"] \arrow[dr, "t"'] & x \arrow[d, dashed, "\exists k"] \\
& y
\end{tikzcd}$$

commutes. This defines a preorder structure on the set Epi_a of epimorphisms with the object a as their domain. We can collapse the proset Epi_a to become a poset $\mathrm{Quot}(a)$. The elements of $\mathrm{Quot}(a)$ are called the quotient object of a.

Example 2.50.

1. Consider the category $\mathbf{Model}_L^T$ and its object $\mathcal{M}$, where T is a positive L-theory. Assume that every epimorphism is an identification. We will show that there is a bijection $\mathrm{Cong}(\mathcal{M}) \cong \mathrm{Quot}(\mathcal{M})$, where we set a congruence to the equivalence class of the corresponding quotient map, $R \mapsto [\mathcal{M} \to \mathcal{M}/R]$. This map is well-defined, because surjective model morphism preserve positive truth by Theorem 1.41(2). The injectivity is clear even

without the assumption that all epimorphisms are surjective. Let $f : \mathcal{M} \to \mathcal{N}$ be an epimorphism. Now by The Fundamental Theorem of Model Morphism 1.46, there exists a unique L-model morphism $\tilde{f} : \mathcal{M}/\ker(f) \to \mathcal{N}$ where the diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r, "f"] \arrow[d, dashed, "q"] & \mathcal{N} \\ \mathcal{M}/\ker(f) \arrow[ur, "\tilde{f}"'] & \end{tikzcd} \tag{2.4}$$

commutes. Additionally, $\mathcal{M}/\ker(f)$ satisfies the theory T and hence is an object in $\mathbf{Model}_L^T$. Since q and f are identifications, it follows that $\tilde{f}$ is a bijective identification and hence an isomorphism. Thus we see the surjectivity of the correspondence $\mathrm{Cong}(\mathcal{M}) \to \mathrm{Quot}(\mathcal{M})$.

2. By the previous argument we see that in the categories **R-Mod** of R-modules and **Grp** of groups, the quotients correspond to the congruences.

3. In the category **Top** fix a space X. Now the congruences of X correspond to the equivalence relations on X, since the quotient maps are able to coinduce the topology on the quotient set $X/\sim$ for any equivalence relation $\sim$ on X. Again we see that there is an injection $\mathrm{Cong} \to \mathrm{Quot}(X)$, but this may not be a surjection. Similarly, as in Example 2.45 (2), consider the discrete space X of two elements and the Sierpinski space S. Now the surjection $X \to S$ does not correspond to any congruence on X, via the canonical map $\mathrm{Cong}(X) \to \mathrm{Quot}(X)$. Thus the cardinality of the set of congruences on X is strictly smaller than the cardinality of quotient objects of X.

4. In a poset category $\mathbf{P}$ the quotient objects of $a \in P$ are those elements $b \in P$ where $a \leq b$.

Next we will define the coimage of a morphism. Dually, to image, coimage of $f : x \to y$ is the smallest quotient object of x through which the morphism f factors through.

Definition 2.51. Let $f : c \to d$ be a morphism in a category $\mathbf{C}$. Let $p : c \to c'$ be an epimorphism in $\mathbf{C}$. Now p is called the coimage of f, if there exists $f' : c' \to d$ where the diagram

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "p"'] & & d \\ & c' \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes and p is terminal among such epimorphisms. In other words, given any epimorphism $p' : c \to c''$ and $f'' : c'' \to d$ where the diagram

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "p'"'] & & d \\ & c'' \arrow[ur, "f''"'] & \end{tikzcd}$$

commutes, then $p \geq p'$. To be specific, there exists a unique $k : c' \to c''$ such that the diagram

$$\begin{tikzcd} c \arrow[rr, "f"] \arrow[dr, "\forall p'"] \arrow[ddr, bend right, "p"'] & & d \\ & \forall c'' \arrow[ur, "\forall f'"] \arrow[d, dotted, "\exists! k"] & \\ & c' \arrow[uur, bend right, "f'"'] & \end{tikzcd} \tag{2.5}$$

commutes.

Example 2.52.

1. As we saw with image, an epimorphism f is always the coimage of itself.

2. In the algebraic categories **Grp** and **R-Mod**, the coimage of an epimorphism $f : X \to Y$ is equivalent with the quotient $q : X \to X/\ker f$.

2.6.3 Sum and product

Here we see that in the great abstractness of a category we can define new interesting objects from given ones. We will talk about product and sum of objects.

Definition 2.53 (Categorical product)**.** Let $\mathbf{C}$ be a category, let a_i and c be objects in $\mathbf{C}$ and let $p_i : c \to a_i$ be a morphism in $\mathbf{C}$ for $i = 1, 2$. The tuple (c, p_1, p_2) is called the product of objects a_1 and a_2, if the following universal property is satisfied: Let t be an object in $\mathbf{C}$ and let $f_i : t \to a_i, i = 1, 2$. Then there exists a unique $\alpha : t \to c$ that makes the diagram

$$\begin{tikzcd}
& \forall t \arrow[d, dotted, "\exists!\alpha"] \arrow[ddl, bend right, "\forall f_1"'] \arrow[ddr, bend left, "\forall f_2"] & \\
& c \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
a_1 & & a_2
\end{tikzcd}$$

commute. We denote the product object by $a_1 \times a_2$. The morphisms p_1 and p_2, called projections, are usually left implicit. The a unique morphism is denoted by (f_1, f_2).

Not all pairs of objects in an arbitrary category have a product. There may be many product objects for a given pair. So we have to justify the use of the article 'the'. This definition of a product can be formulated as a terminal object in a suitable category and hence all products over the same pair are the same up to a unique isomorphism:

We define a new category of cones $\mathbf{Cone}_{(a_1,a_2)}$ over the pair (a_1, a_2).

1. Objects are tuples $(c, f : c \to a_1, g : c \to a_2)$, where c is an object and f and g are morphisms in $\mathbf{C}$.

2. Morphisms $\alpha : (c, f_1, f_2) \to (d, g_1, g_2)$ is a morphism $\alpha : c \to d$ where the diagram

$$\begin{tikzcd}
& c \arrow[d, "\alpha"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& d \arrow[dl, "g_1"] \arrow[dr, "g_2"'] & \\
a_1 & & a_2
\end{tikzcd}$$

commutes.

3. Given morphisms $(c, f_1, f_2) \xrightarrow{\alpha} (d, g_1, g_2) \xrightarrow{\beta} (e, h_1, h_2)$, hence the there is the diagram

$$\begin{tikzcd}
& c \arrow[d, "\alpha"] \arrow[dddl, bend right, "f_1"'] \arrow[dddr, bend left, "f_2"] & \\
& d \arrow[d, "\beta"] \arrow[ddl, bend right, "g_1"'] \arrow[ddr, bend left, "g_2"] & \\
& e \arrow[dl, "h_1"] \arrow[dr, "h_2"'] & \\
a_1 & & a_2
\end{tikzcd} .$$

Now we define the composition as the morphism $(c, f_1, f_2) \xrightarrow{\beta\circ\alpha} (e, h_1, h_2)$

$$\begin{tikzcd}
& c \arrow[d, "\beta\alpha"] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\
& e \arrow[dl, "h_1"] \arrow[dr, "h_2"'] & \\
a_1 & & a_2
\end{tikzcd}$$

The composite diagram commutes. Thus the category $\mathbf{Cone}_{(a_1,a_2)}$ is well-defined. The product object of the objects a_1 and a_2 is the terminal object in the cone category.

On one hand, the universal property of product yields us a function

$$\mathbf{C}(x, a_1) \times \mathbf{C}(x, a_2) \to \mathbf{C}(x, a_1 \times a_2), (f, g) \mapsto \alpha_{f,g},$$

where the outer product is the Cartesian product. On the other, hand there is a function

$$\mathbf{C}(x, a_1 \times a_2) \to \mathbf{C}(x, a_1) \times \mathbf{C}(x, a_2), \alpha \mapsto (pr_1 \circ \alpha, pr_2 \circ \alpha).$$

Directly from the universal property we see that these functions are each other's inverses. Hence there is a bijection $\mathbf{C}(x, a_1 \times a_2) \cong \mathbf{C}(x, a_1) \times \mathbf{C}(x, a_2)$.

Example 2.54.

1. Let X and Y be sets. In the category **Set** the tuple $(X \times Y, pr_1, pr_2)$ is the product object of the objects X and Y, where the product is the standard Cartesian product and the functions are the usual projection maps. Let (Z, f, g) be any other solution over the pair (X, Y). We are required to show that there exists a unique function $\alpha : (Z, f, g) \to (X \times Y, pr_1, pr_2)$.

$$\begin{tikzcd} & Z \arrow[d, dashed, "\alpha"] \arrow[ddl, bend right, "f"'] \arrow[ddr, bend left, "g"] & \\ & X \times Y \arrow[dl, "pr_1"] \arrow[dr, "pr_2"'] & \\ X & & Y \end{tikzcd}$$

   Uniqueness: Assume there exists such an α. Let $z \in Z$. Now $\alpha(z) = (x, y)$ where $x \in X$ and $y \in Y$. By projecting, we see that $f(z) = x$ and $g(z) = y$. Therefore $\alpha(z) = (f(z), g(z))$ for $z \in Z$.

   Existence: We get the wanted α by defining it by the previous equation.

2. Let $(\mathbf{C}, F)$ be a topological category. Let a and b be objects of $\mathbf{C}$. Consider the the product $Fa \times Fb$ in the category of sets. Now the projections F-induce a structure on $Fa \times Fb$. Directly from theomre 2.36 we see that the induced structure defines the categorical product of $a \times b$ in $\mathbf{C}$. Thus the category of topological spaces has product objects.

3. Let $\mathcal{M}$ and $\mathcal{N}$ be objects in the algebraic category $\mathbf{Model}_L^T$. Now there exists the product model $\mathcal{M} \times \mathcal{N}$ which also satisfies the theory T by Theorem 1.42. We will show that $(\mathcal{M} \times \mathcal{N}, pr_1, pr_2)$ is the product object. Let $\phi_1 : \mathcal{X} \to \mathcal{M}$ and $\phi_2 : \mathcal{X} \to \mathcal{N}$ be model morphisms in $\mathbf{Model}_L^T$. Denote the corresponding universes by M, N and X of $\mathcal{M}, \mathcal{N}$ and $\mathcal{X}$, respectively. Now there exists a unique function $\phi : X \to M \times N$ where the diagram

$$\begin{tikzcd} & X \arrow[d, dashed, "\phi"] \arrow[ddl, bend right, "\phi_1"'] \arrow[ddr, bend left, "\phi_2"] & \\ & M \times N \arrow[dl, "pr_1"] \arrow[dr, "pr_2"'] & \\ M & & N \end{tikzcd}$$

   commutes. It remains to show that ϕ is a model morphism. For a constant symbol c in L,

$$\phi(c^{\mathcal{X}}) = (\phi_1(c^{\mathcal{X}}), \phi_2(c^{\mathcal{X}})) = (c^{\mathcal{M}}, c^{\mathcal{N}}) = c^{\mathcal{M} \times \mathcal{N}}.$$

   Next fix a function symbol f in L. Then

$$\begin{aligned} \phi f^{\mathcal{X}} &= (\phi_1 f^{\mathcal{X}}, \phi_2 f^{\mathcal{X}}) \\ &= (f^{\mathcal{M}} \phi_{1*}, f^{\mathcal{N}} \phi_{2*}) \\ &= f^{\mathcal{M} \times \mathcal{N}} \phi_*. \end{aligned}$$

   Lastly, fix a relation symbol R in L and $k = \text{ord}(R)$. Let $(x_1, \ldots, x_k) \in R^{\mathcal{X}}$. Now

$$\phi_*(x_1, \ldots, x_k) = ((\phi_1(x_1), \phi_2(x_1)), \ldots (\phi_1(x_k), \phi_2(x_k))) \in R^{\mathcal{M} \times \mathcal{N}}.$$

   Hence ϕ is a model morphism.

4. If P is a proset, then the product of two objects is the infimum of the two. So in $\mathcal{P}(X)$ (with ordering defined by the subset relation) the product of two sets is the intersection. In the linearly ordered set of real numbers, the product of objects is their minimum. A particularly interesting poset is $(\mathbb{N}, |)$ where $|$ is the divides relation. Here taking the product is the same as finding the greatest common divisor.

5. The category of small categories **Cat** has products: Let $\mathbf{C}$ and $\mathbf{D}$ be small categories. We define a category $\mathbf{C} \times \mathbf{D}$ whose objects are of the form (c, d) where c and d are objects of $\mathbf{C}$ and $\mathbf{D}$, respectively. A morphisms $(c, d) \to (c', d')$ in $\mathbf{C} \times \mathbf{D}$ is a pair (f, g) where $f : c \to c'$ and $g : d \to d'$. The composition is defined componentwise. The canonical projections define the projection functors.

Properties of products

If we assume that a category has a product object for every pair of objects and a terminal object, we will show here that the product operation satisfies abelian monoid structure up to canonical isomorphisms. It will be useful later on to see that some diagrams built from these canonical isomorphisms always commute. The commutative diagrams and the canonical isomorphism justify the treatment of product objects as if they were elements in an abelian monoid even though strictly speaking $(a \times b) \times c \neq a \times (b \times c)$.

Definition 2.55. Let $a_1 \xrightarrow{f} b_1$ and $a_2 \xrightarrow{g} b_2$ be morphisms in $\mathbf{C}$. Assume the product objects $a_1 \times a_2$ and $b_1 \times b_2$ exist. We define the product morphism $f_1 \times f_2 : a_1 \times a_2 \to b_1 \times b_2$ as (fpr_1, gpr_2), which can be seen from the following commutative diagram:

$$\begin{tikzcd}
& a_1 \times a_2 \arrow[ld, bend right] \arrow[rd, bend left] \arrow[d, dotted, "(fpr_1, gpr_2)"] & \\
a_1 \arrow[d, "f"'] & b_1 \times b_2 \arrow[ld] \arrow[rd] & a_2 \arrow[d, "g"] \\
b_1 & & b_2
\end{tikzcd}$$

Notice that the projection maps were left as implicit.

Theorem 2.56. *Let $a_i \xrightarrow{f_i} b_i, b_i \xrightarrow{g_i} c_i$ be morphisms in a category $\boldsymbol{C}$ for $i = 1, 2$. Assume that the products $a_1 \times a_2, b_1 \times b_2$ and $c_1 \times c_2$ exist in $\boldsymbol{C}$. Then*

$$(g_1 \times g_2) \circ (f_1 \times f_2) = g_1 f_1 \times g_2 f_2.$$

Additionally, the product of identities is an identity.

Proof. Notice that $g_1 f_1 \times g_2 f_2$ is the unique morphism $t : a_1 \times a_2 \to c_1 \times c_2$ where $pr_1 t = g_1 f_1 pr_1$ and $pr_2 t = g_2 f_2 pr_2$. Now

$$pr_1 \circ (g_1 \times g_2) \circ (f_1 \times f_2) = g_1 pr_1 \circ (f_1 \times f_2) = g_1 f_1 pr_1.$$

The other equation is shown to be true similarly. Lastly, $pr_i \circ (id_{a_1} \times id_{a_2}) = pr_i$ for $i = 1, 2$. By uniqueness, $id_{a_1} \times id_{a_2} = id_{a_1 \times a_2}$. $\square$

Theorem 2.57 (Associator, unitor and symmetor)**.** *Let $\boldsymbol{C}$ be a category with binary product objects and a terminal object* 1. *Let a, b and c be objects in $\boldsymbol{C}$. Then there exist canonical isomorphisms*

- $\alpha_{a,b,c} : (a \times b) \times c \cong a \times (b \times c)$ *called the associator.*
- $L_a : 1 \times a \cong a$ *and* $R_a : a \times 1 \cong a$ *called the left and right unitors.*
- $s_{a,b} : a \times b \cong b \times a$ *called the symmetor.*

Proof.

- The associator $\alpha_{a,b,c}$ is defined to be the morphism $(p \circ p, (q \circ p, q))$ where p refers to the projection to the left component and q to the projection on the right component. Notice that there is abuse of notation, where we refer to two potentially different projection maps with the same symbol.

$$\begin{tikzcd}
 & (a \times b) \times c \arrow[dl, "p"'] \arrow[d, dashed, "(q\circ p, q)"] \arrow[ddr, bend left, "q"] & \\
a \times b \arrow[d, "q"] & b \times c \arrow[dl, "p"] \arrow[dr, "q"'] & \\
b & & c
\end{tikzcd}
\qquad
\begin{tikzcd}
 & (a \times b) \times c \arrow[dl, bend right, "p"'] \arrow[d, dashed, "(p\circ p, (q\circ p, q))"] \arrow[ddr, bend left, "(q\circ p, q)"] & \\
a \times b \arrow[d, "p"] & a \times (b \times c) \arrow[dl, "p"] \arrow[dr, "q"'] & \\
a & & b \times c
\end{tikzcd}$$

Similarly, we define the potential inverse as $((p, p \circ q), q \circ q)$. We need to check that

$$(pp, (qp, q))((p, pq), qq) = id$$

and

$$((p, pq), qq))(pp, (qp, q)) = id.$$

The cases are similar so we will check only the first. Notice that $id = (p, (pq, qq)) : a \times (b \times c) \to a \times (b \times c)$:

$$\begin{aligned}
p(pp, (qp, q))((p, pq), qq) &= pp((p, pq), qq) \\
&= p,
\end{aligned}$$

$$\begin{aligned}
pq(pp, (qp, q))((p, pq), qq) &= p(qp, q)((p, pq), qq) \\
&= qp((p, pq), qq) \\
&= q(p, pq) \\
&= pq
\end{aligned}$$

and

$$\begin{aligned}
qq(pp, (qp, q))((p, pq), qq) &= q((p, pq), qq) \\
&= qq.
\end{aligned}$$

- Since 1 is a terminal object there exists the following cone:

$$\begin{tikzcd}
 & a \arrow[dl, "!"'] \arrow[dr, "id_a"] & \\
1 & & a
\end{tikzcd}$$

which is terminal. Thus a is the product object of a and 1. Hence there exists a unique suitable morphism from $1 \times a$ to a and it is an isomorphism. The case for right unitor is similar.

- The symmetor is defined as (q, p) which is an isomorphism.

□

Theorem 2.58 (Coherence laws and naturality)**.** *Let* $\boldsymbol{C}$ *be a category with binary products and a terminal object denoted* 1*. Then the following diagrams commute in* $\boldsymbol{C}$ *via the canonical morphisms:*

1. *Coherence of associator with itself, commutation of the pentagon diagram:*

$$\begin{tikzcd}
 & (a \times b) \times (c \times d) \arrow[dr] & \\
((a \times b) \times c) \times d \arrow[ur] \arrow[d] & & a \times (b \times (c \times d)) \\
(a \times (b \times c)) \times d \arrow[rr] & & a \times ((b \times c) \times d) \arrow[u]
\end{tikzcd}$$

2. *Coherence between the associator and unitor, commutation of the triangle diagram:*

$$
\begin{tikzcd}
(a \times 1) \times b \arrow[rr] \arrow[dr] & & a \times (1 \times b) \arrow[dl] \\
& a \times b &
\end{tikzcd}
$$

3. *Symmetors coherence with unitor, associator and itself:*

$$
\begin{tikzcd}
a \times 1 \arrow[rr] \arrow[dr] & & 1 \times a \arrow[dl] \\
& a &
\end{tikzcd}
\qquad
\begin{tikzcd}
(a \times b) \times c \arrow[r] \arrow[d] & (b \times a) \times c \arrow[d] \\
a \times (b \times c) \arrow[d] & b \times (a \times c) \arrow[d] \\
(b \times c) \times a \arrow[r] & b \times (c \times a)
\end{tikzcd}
\qquad
\begin{tikzcd}
& b \times a \arrow[dr] & \\
a \times b \arrow[ur] \arrow[rr, "id_{a \times b}"'] & & a \times b
\end{tikzcd}
$$

4. *Naturality conditions for the associator, the unitors and the symmetor: Let $f : a \to a', g : b \to b'$ and $h : c \to c'$ morphisms in $\boldsymbol{C}$. Then the following diagrams commute:*

$$
\begin{tikzcd}
(a \times b) \times c \arrow[r] \arrow[d, dashed, "(f \times g) \times h"] & a \times (b \times c) \arrow[d, dashed, "f \times (g \times h)"] \\
(a' \times b') \times c' \arrow[r] & a' \times (b' \times c')
\end{tikzcd}
\quad
\begin{tikzcd}
1 \times a \arrow[r] \arrow[d, dashed, "id \times f"] & a \arrow[d, dashed, "f"] \\
1 \times b \arrow[r] & b
\end{tikzcd}
\quad
\begin{tikzcd}
a \times 1 \arrow[r] \arrow[d, dashed, "f \times id"] & a \arrow[d, dashed, "f"] \\
b \times 1 \arrow[r] & b
\end{tikzcd}
\quad
\begin{tikzcd}
a \times b \arrow[r] \arrow[d, dashed, "f \times g"] & b \times a \arrow[d, dashed, "g \times f"] \\
a' \times b' \arrow[r] & b' \times a'
\end{tikzcd}
$$

Proof. The calculations are similar to the ones the proof of Theorem 2.57. ☐

When one generalizes the concept of a product these coherence laws are naturally required. Mac Lane in his book "Categories for the Working Mathematician"[4] shows that from these coherence laws one can deduce that all finite diagrams made out of product, associator and unitor will yield a commutative diagram in a quite general setting. The result holds for example to tensor product of vector spaces.

The definition of product was given in an arbitrary category. This hints about it is dual concept. The dual of product is called sum.

Definition 2.59. Let a_1 and a_2 be objects in a category $\mathbf{C}$. Then the sum of the objects a_1 and a_2 is the product in the category $\mathbf{C}^{op}$. The sum is denoted by $a_1 + a_2$ and the corresponding morphisms are called injections.

The sum object $a_1 + a_2$ consists of the following information: There are morphisms $i_1 : a_1 \to a_1 + a_2$ and $i_2 : a_2 \to a_1 + a_2$

$$
\begin{tikzcd}
a_1 \arrow[dr, "i_1"'] & & a_2 \arrow[dl, "i_2"] \\
& a_1 + a_2 &
\end{tikzcd}
.
$$

Given any other such diagram in $\mathbf{C}$

$$
\begin{tikzcd}
a_1 \arrow[dr, "f"'] & & a_2 \arrow[dl, "g"] \\
& x &
\end{tikzcd}
$$

there exists a unique morphism $\alpha : a_1 + a_2 \to x$ such that the diagram

$$
\begin{tikzcd}
a_1 \arrow[dr, "i_1"] \arrow[ddr, bend right, "\forall f"'] & & a_2 \arrow[dl, "i_2"'] \arrow[ddl, bend left, "\forall g"] \\
& a_1 + a_2 \arrow[d, dashed, "\exists! \alpha"] & \\
& \forall x &
\end{tikzcd}
$$

commutes. The morphism α is denoted by (f, g).

Again there is a bijection $\mathbf{C}(a_1 + a_2, x) \cong \mathbf{C}(a_1, x) \times \mathbf{C}(a_2, x)$.

Example 2.60.

1. Categorical sums in **Set** are disjoint unions. To be more specific given sets X and Y, there exists injections $X \hookrightarrow X \sqcup Y$ and $Y \hookrightarrow X \sqcup Y$, where $X \sqcup Y := \{0\} \times X \cup \{1\} \times Y$. This information defines the categorical sum in **Set**.

2. In **Top** the same construction defines categorical sum, when the injections coinduce the topology on the disjoint union. This generalizes to all topological categories.

3. The category **Mon** is an interesting case. Here the sums are free sums of monoids: Let M and N be monoids. Then their free product is $M \star N$ whose elements are equivalence classes finite sequences of elements in $M \sqcup N$ where sequences are identified, if we can get from one sequence to an other by composing nearby elements when possible. For example

$$[(a_1, ..., a_{k-1}, a_k, a_{k+1}, a_{k+2}, ..., a_n)] = [(a_1, ..., a_{k-1}, a_k a_{k+1}, a_{k+2} ..., a_n)],$$

if $a_k, a_{k+1} \in M$ or $a_k, a_{k+1} \in N$. The equivalence relation is the smallest equivalence relation with the above property. The monoid operation is defined by concatenating these sequences. The monoids M and N can be injected to the free sum by $a \mapsto [(a)]$.

4. With posets P the sum of object of $a, b \in P$, perhaps not surprisingly but satisfyingly, the supremum of $\{a, b\}$, if it exists. In the poset $\mathcal{P}(X)$, where X is a set, supremums are unions, in linearly ordered real numbers maximum and in $(\mathbb{N}, |)$ the least common multiple.

5. The category **Cat** has coproducts: Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We define $\mathbf{C} + \mathbf{D}$ to be the category whose set of objects is the disjoint unions of objects of $\mathbf{C}$ and $\mathbf{D}$. Similarly for the set of morphisms. The domains and codomains of morphisms in $\mathbf{C}$ and $\mathbf{D}$ are kept the same in $\mathbf{C} + \mathbf{D}$ and similarly for composition. The canonical injections define the coproduct structure on $\mathbf{C} + \mathbf{D}$.

2.7 Exponentiation

We use the notation X^Y to denote the set of functions from the set Y to the set X in the category **Set** of sets. This notion can be generalized to an arbitrary category. We begin our search by finding a suitable universal property. There exists the evaluation map $X^Y \times Y \xrightarrow{ev} X$, $(f, x) \mapsto f(x)$. We can think that the diagram

$$X^Y \times Y \xrightarrow{ev} X$$

is an object in a suitable category. We will find that this diagram is terminal in the wisely chosen category. The universal property of the diagram is the following: Given any set Z and a function $Z \times Y \xrightarrow{f} X$, then there exists a unique morphism $Z \xrightarrow{\tilde{f}} X^Y$ where the following diagram commutes:

$$\begin{tikzcd}
X^Y \times Y \arrow[r, "ev"] & X \\
Z \times Y \arrow[u, dotted, "\tilde{f} \times id_Y"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Uniqueness: Assuming there exists such a morphism $\tilde{f}$, we get that, for $z \in Z$ and $y \in Y$,

$$f(z, y) = ev(\tilde{f}(z), y) = (\tilde{f}(z))(y).$$

Hence by the arbitrariness of y, there exists the function $\tilde{f}(z)$ completely defined. Since z was arbitrary there exists the function $\tilde{f}$ is uniquely specified. Therefore $\tilde{f}$ is unique. Existence is now clear by constructing the function from the previous considerations.

We will define the category in which our exponential object is terminal. The category of exponential solutions $\mathbf{E}_{(X,Y)}$ is as follows:

1. Objects are pairs $(Z, f : Z \times Y \to X)$ where Z is a set and f is a function:

$$Z \times Y \xrightarrow{f} X$$ [5]

2. The morphism $(W, g) \to (Z, f)$ consists of a function $\alpha : W \to Z$ where the following diagram commutes:

$$\begin{tikzcd} Z \times Y \arrow[r, "f"] & X \\ W \times Y \arrow[u, dashed, "\alpha \times id_Y"] \arrow[ur, "g"'] & \end{tikzcd}$$

3. The composition is defined as usual. By Theorem 2.56, the composition is well-defined.

The object $(X^Y, ev : X^Y \times Y \to X)$ is terminal in the category $\mathbf{E}_{(X,Y)}$. This motivates us to give the definition of exponential objects as follows:

Definition 2.61 (Exponential object)**.** Let a and b be objects in a category $\mathbf{C}$. Assume that for all objects c in $\mathbf{C}$ the product $c \times a$ exists. The pair $(c, f : c \times a \to b)$ is called the exponential object of the pair (a, b), denoted by b^a and $[a, b]$, if it is terminal in the category of $\mathbf{E}_{a,b}$, where the definition is similar as above:

1. The objects are pairs $(c, f : c \times a \to b)$ where c is an object and f is a morphism in $\mathbf{C}$. We leave the projections implicit in the notation $c \times a$.

2. The morphism $\alpha : (c, f) \to (d, g)$ is a morphism $\alpha : c \to d$ where the following diagram commutes:

$$\begin{tikzcd} d \times a \arrow[r, "g"] & b \\ c \times a \arrow[u, dashed, "\alpha \times id_a"] \arrow[ur, "f"'] & \end{tikzcd}.$$

3. The composition is defined by the composition in $\mathbf{C}$. By Theorem 2.56, the composition is well-defined.

The object a is called an exponentiable object, if the exponential object b^a exists for all objects b in $\mathbf{C}$.

Notice that similarly as for products and sums we get a bijection $\mathbf{C}(c \times a, b) \cong \mathbf{C}(c, b^a)$. This correspondence is called (exponential) transposition where to $f : c \times a \to b$ is associated the map $\tilde{f} : c \to b^a$. The inverse operation is called (exponential) application. If $\mathbf{C}$ has a terminal object 1, binary products and the exponential b^a exists for some objects a and b in $\mathbf{C}$, then $\mathbf{C}(1, b^a) \cong \mathbf{C}(1 \times a, b) \cong \mathbf{C}(a, b)$.

Example 2.62.

1. As we saw earlier, the category **Set** has exponential objects over any pair of sets.

2. In the category **Top**, of topological spaces, the exponentiation Y^X is not defined for all spaces X and Y. In the paper by Escardo and Heckmann "Topologies on spaces of continuous functions"[3] it is shown that a Hausdorff space is exponentiable, if and only if it is locally compact. Because not all Hausdorff spaces are locally compact, it follows that **Top** does not have all exponents.

3. If a category $\mathbf{C}$ is pointed and $\mathbf{C}$ has an exponential b^a, then $\mathbf{C}(0, b^a) \cong \mathbf{C}(a, b)$. Thus there exists only a single morphism from a to b. Therefore in the category **Mon** of monoids, exponentials exists very rarely.

[5] Since the notation $Z \times Y$ implicitly encodes the projection morphisms, objects in $\mathbf{E}_{(X,Y)}$ contain the information of these projections.

4. In a poset category $\mathcal{P}(X)$ the exponential B^A of sets $A, B \subset X$ is the set $(X \setminus A) \cup B$. First we recognize that $((X \setminus A) \cup B) \cap A \subset B$. Assume that $C \cap A \subset B$. Now we see that

$$\begin{aligned} C &= (C \setminus A) \cup (C \cap A) \\ &\subset (X \setminus A) \cup B. \end{aligned}$$

   This generalizes to any Boolean algebra: Let x, y be elements in a Boolean algebra B. Then

$$y^x = \neg x \vee y =: x \to y.$$

5. Let P and Q be objects in the category **Proset** of prosets. We define a new proset

$$Q^P := \{f : P \to Q \mid f(a) \leq f(b) \text{ for all } a, b \in P, a \leq b\},$$

   where we define $f \leq g$, if $f(a) \leq g(a)$ for all $a \in P$ and $f, g \in Q^P$. The evaluation map $ev : Q^P \times P \to Q$ is increasing, since, if $(f, a) \leq (g, b)$, then

$$ev(f, a) = f(a) \leq g(a) \leq g(b) = ev(g, b).$$

   Let $f : X \times P \to Q$ be an increasing map, where X is a proset. The map $\tilde{f}$, defined as in the case of **Set**, is increasing. Let $x, y \in X$ and assume $x \leq y$. We need to show that $\tilde{f}(x) \leq \tilde{f}(y)$. Let $p \in P$. Since $(x, p) \leq (y, p)$, it holds that

$$\tilde{f}(x)(p) = f(x, p) \leq f(y, p) = \tilde{f}(y)(p).$$

   Hence $\tilde{f}(x) \leq \tilde{f}(y)$. Thus $\tilde{f}$ is an increasing function. This proves that the category **Proset** has exponentials.

Having exponentials in a category makes it possible to internalize category theory within the studied category. For example, there exists the composition function

$$\circ : \mathbf{Set}(Y, Z) \times \mathbf{Set}(X, Y) \to \mathbf{Set}(X, Z)$$

from the definition of the category **Set**. This composition is also a morphism in **Set**. We may ask, how often this happens. Can we change from outside perspective of a category to the internal side of it? Categories where this is reasonably possible are called Cartesian closed categories:

Definition 2.63. We say that a category $\mathbf{C}$ is called Cartesian closed, if it has a terminal object, binary products and all exponentials.

Theorem 2.64 (Internal Composition Theorem)**.** *Let $\mathbf{C}$ be a Cartesian closed category, let a, b, c be objects and let $1 \xrightarrow{x} a \xrightarrow{f} b \xrightarrow{g} c$ be morphisms in $\mathbf{C}$. Then the following assertions hold:*

1. *Denote by $\tilde{f}$ the exponential transposition of the map of $f pr_2$. Then $ev \circ (\tilde{f}, x) = f \circ x$.*

$$\begin{tikzcd}
b^a \times a \arrow[rr, "ev"] & & b \\
1 \times a \arrow[u, dotted, "\tilde{f} \times id_a"] \arrow[r, "pr_2"] & a \arrow[ur, "f"] &
\end{tikzcd}$$

2. *There exists an internal composition morphism $comp : c^b \times b^a \to c^a$.*

3. *The internal composition corresponds to the categorical composition*

$$comp \circ (\tilde{g}, \tilde{f}) = \widetilde{g \circ f},$$

   where the exponential transposition is done by the construction used in part 1.

4. *With respect to the internal composition the internal hom-set a^a has a unique neutral element $\widetilde{id}_a$.*

5. *The internal composition comp is associative.*

Proof. 1. The diagram

$$\begin{tikzcd}
b^a \times a \\
1 \times a \arrow[u, "\tilde{f} \times id_a"] \\
1 \times 1 \arrow[u, "id_1 \times x"] \\
1 \arrow[u, "pr_1^{-1}"] \arrow[uuu, bend left, "(\tilde{f},x)"]
\end{tikzcd}$$

commutes, since

$$pr_1((\tilde{f} \times id_a) \circ (id_1 \times x) \circ pr_1^{-1}) = \tilde{f} pr_1 \circ (id_1 \times x) \circ pr_1^{-1} = \tilde{f} id_1 pr_1 pr_1^{-1} = \tilde{f}.$$

Since projections of the object 1×1 are the same, we get

$$pr_2 \circ ((\tilde{f} \times id_a) \circ (id_1 1 \times x) \circ pr_1^{-1}) = id_a pr_2 \circ (id_1 \times x) \circ pr_2^{-1} = x pr_2 pr_2^{-1} = x.$$

Thus $(\tilde{f}, x) = (\tilde{f} \times id_a) \circ (id_1 \times x) \circ pr_1^{-1}$. It remain to see that $ev \circ (\tilde{f}, x) = fx$, which holds since

$$ev \circ (\tilde{f}, x) = ev \circ (\tilde{f} \times id_a) \circ (id_1 \times x) \circ pr_1^{-1} = f pr_2 \circ (id_1 \times x) \circ pr_1^{-1} = f x pr_2 pr_2^{-1} = fx$$

2. The morphism $comp : c^b \times b^a \to c^a$ is the exponential transposition of morphism of $ev \circ (id_{c^b} \times ev) \circ \alpha$, where as a reminder $\alpha = (pr_1 pr_1, (pr_2 pr_1, pr_2))$ with suitable projections. Thus the diagram

$$\begin{tikzcd}
c^a \times a \arrow[rr, "ev"] & & c \\
(c^b \times b^a) \times a \arrow[u, "\mathrm{comp} \times id_a"] \arrow[r, "\alpha"] & c^b \times (b^a \times a) \arrow[r, "id_{c^b} \times ev"] & c^b \times b \arrow[u, "ev"]
\end{tikzcd}$$

commutes.

3. We need to show the compatibility condition that

$$\mathrm{comp} \circ (\tilde{g}, \tilde{f}) = \widetilde{g \circ f}.$$

It suffices to show that the diagram

$$\begin{tikzcd}
b^a \times a \arrow[rr, "ev"] & & c \\
1 \times a \arrow[u, "(\mathrm{comp} \circ (\tilde{g},\tilde{f})) \times id_a"] \arrow[r, "pr_2"] & a \arrow[r, "f"] & b \arrow[u, "g"]
\end{tikzcd}$$

commutes. First we show that the diagram

$$\begin{tikzcd}
c^b \times b \arrow[rr, "id_{c^b \times b}"] & & c^b \times b \\
1 \times a \arrow[u, "\tilde{g} \times f"] \arrow[r, "(\tilde{g},\tilde{f}) \times id_a"] & (c^b \times b^a) \times a \arrow[r, "\alpha"] & c^b \times (b^a \times a) \arrow[u, "id_{c^b} \times ev"]
\end{tikzcd}$$

commutes. We see this by

$$\begin{aligned}
pr_1 \circ (id_{c^b} \times ev) \circ \alpha \circ ((\tilde{g}, \tilde{f}) \times id_a) &= pr_1 \circ \alpha \circ ((\tilde{g}, \tilde{f}) \times id_a) \\
&= pr_1 pr_1 \circ ((\tilde{g}, \tilde{f}) \times id_a) \\
&= \tilde{g} pr_1
\end{aligned}$$

and

$$\begin{aligned} pr_2 \circ (id_{c^b} \times ev) \circ \alpha \circ ((\tilde{g}, \tilde{f}) \times id_a) &= evpr_2 \circ \alpha \circ ((\tilde{g}, \tilde{f}) \times id_a) \\ &= ev \circ (pr_2 pr_1, pr_2) \circ ((\tilde{g}, \tilde{f}) \times id_a) \\ &= ev \circ (\tilde{f} pr_1, pr_2) \\ &= ev \circ (\tilde{f} \times id_a) \\ &= f pr_2. \end{aligned}$$

Hence

$$\begin{aligned} ev \circ ((\mathrm{comp} \circ (\tilde{g}, \tilde{f})) \times id_a) &= ev \circ (\mathrm{comp} \times id_a) \circ ((\tilde{g}, \tilde{f}) \times id_a) \\ &= ev \circ (id_{c^b} \times ev) \circ \alpha \circ ((\tilde{g}, \tilde{f}) \times id_a) \\ &= ev \circ (\tilde{g} \times f) \\ &= ev \circ (\tilde{g} \times id_a) \circ (id_1 \times f) \\ &= g pr_2 \circ (id_1 \times f) \\ &= g f pr_2. \end{aligned}$$

4. Let $e : 1 \to a^a$. We need to show that $e = \tilde{id_a}$, if and only if for any objects b, c and morphisms $f : 1 \to b^a, g : 1 \to a^c$ the diagrams

$$\begin{tikzcd} b^a \times a^a \arrow[r, "\mathrm{comp}"] & b^a \\ 1 \arrow[u, "{(f,e)}"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} a^a \times a^c \arrow[r, "\mathrm{comp}"] & a^c \\ 1 \arrow[u, "{(e,g)}"] \arrow[ur, "g"'] & \end{tikzcd}$$

commute. Assume first that $e = \tilde{id_a}$ and $f : 1 \to b^a$ and $g : 1 \to a^c$ are morphisms in $\mathbf{C}$. Denote by f' and g' the a unique morphisms that make the following diagrams commute

$$\begin{tikzcd} b^a \times a \arrow[r, "ev"] & b \\ 1 \times a \arrow[u, "f \times id_a"] & a \arrow[u, "f'"'] \arrow[l, "pr_2^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} a^c \times c \arrow[r, "ev"] & a \\ 1 \times c \arrow[u, "g \times id_c"] & c \arrow[u, "g'"'] \arrow[l, "pr_2^{-1}"] \end{tikzcd}$$

Now

$$\mathrm{comp} \circ (f, e) = \mathrm{comp} \circ (\tilde{f'}, \tilde{id_a}) = \widetilde{f' id_a} = \tilde{f'} = f$$

and similarly $\mathrm{comp} \circ (e, g) = g$.

Assume next that $\mathrm{comp} \circ (e, g) = g$ for all $g : 1 \to c^a$ and c in $\mathbf{C}$. Thus we may choose $g = \tilde{id_a}$. Thus

$$\widetilde{id_a} = g = \mathrm{comp}(e, g) = \mathrm{comp} \circ (\tilde{e'}, \tilde{id_a}) = \widetilde{e' id_a} = \tilde{e'} = e,$$

where e' is similarly defined as f'.

5. Lastly, we will show that the diagram

$$\begin{tikzcd} (d^c \times c^b) \times b^a \arrow[r, "\mathrm{comp} \times id_{b^a}"] \arrow[d, "\alpha"] & d^b \times b^c \arrow[r, "\mathrm{comp}"] & d^a \\ d^c \times (c^b \times b^a) \arrow[r, "id_{d^c} \times \mathrm{comp}"] & d^c \times c^a \arrow[ur, bend right, "\mathrm{comp}"'] & \end{tikzcd} \tag{2.6}$$

commutes. Now

$$\begin{aligned} \mathrm{comp} \circ (id_{d^c} \times \mathrm{comp}) \circ \alpha &= \mathrm{comp} \circ (pr_1, \mathrm{comp} \circ pr_2) \circ (pr_1 pr_1, (pr_2 pr_1, pr_2)) \\ &= \mathrm{comp} \circ (pr_1 pr_1, \mathrm{comp} \circ (pr_2 pr_1, pr_2)) \\ &= \mathrm{comp} \circ (pr_1 pr_1, \mathrm{comp} \circ (pr_2 \times id_{b^a})). \end{aligned}$$

It suffices to show that the following diagram commutes for either morphism, from diagram 2.6, of the form

$$T : (d^c \times c^b) \times b^a \to d^a.$$

$$
\begin{tikzcd}
d^a \times a \arrow[rrr, "ev"] & & & d \\
((d^c \times c^b) \times b^a) \times a \arrow[u, "T \times id_a"] \arrow[r, "\beta"] & d^c \times (c^b \times (b^a \times a)) \arrow[r, "id_{d^c} \times (id_{c^b} \times ev)"] & d^c \times (c^b \times b) \arrow[r, "id_{d^c} \times ev"] & d^c \times c \arrow[u, "ev"']
\end{tikzcd}
\tag{2.7}
$$

where $\beta = (pr_1 pr_1 pr_1, (pr_2 pr_1 pr_1, (pr_2 pr_1, pr_2)))$. Here we see that diagram 2.7 commutes for $T = \text{comp}_{(a,b,d)} \circ (\text{comp}_{(b,c,d)} \times id_{b^a})$

$$
\begin{aligned}
ev \circ (T \times id) &= ev \circ ((\text{comp} \circ (\text{comp} \times id)) \times id) \\
&= ev \circ (\text{comp} \times id) \circ ((\text{comp} \times id) \times id) \\
&= ev \circ (id \times ev) \circ \alpha \circ ((\text{comp} \times id) \times id) \\
&= ev \circ (id \times ev) \circ (\text{comp} \times (id \times id)) \circ \alpha \\
&= ev \circ (\text{comp} \times ev) \circ \alpha \\
&= ev \circ (\text{comp} \times id) \circ (id \times ev) \circ \alpha \\
&= ev \circ (id \times ev)\alpha((id \times id) \times ev))\alpha \\
&= ev \circ (id \times ev)(id \times (id \times ev))\alpha\alpha \\
&= ev \circ (id \times ev)(id \times (id \times ev))\beta.
\end{aligned}
$$

We get the same equation with $T = \text{comp}_{a,c,d} \circ (id_{d^c} \times \text{comp}_{a,b,c}) \circ \alpha$, since

$$
\begin{aligned}
ev \circ (T \times id) &= ev \circ ((\text{comp} \circ (id \times \text{comp}) \circ \alpha) \times id) \\
&= ev \circ (\text{comp} \times id) \circ ((id \times \text{comp}) \times id) \circ (\alpha \times id) \\
&= ev \circ (id \times ev) \circ \alpha \circ ((id \times \text{comp}) \times id) \circ (\alpha \times id) \\
&= ev \circ (id \times ev) \circ (id \times (\text{comp} \times id)) \circ \alpha \circ (\alpha \times id) \\
&= ev \circ (id \times (ev \circ (\text{comp} \times id))) \circ \alpha \circ (\alpha \times id) \\
&= ev \circ (id \times (ev \circ (id \times ev) \circ \alpha)) \circ \alpha \circ (\alpha \times id) \\
&= ev \circ (id \times ev) \circ (id \times (id \times ev)) \circ (id \times \alpha) \circ \alpha \circ (\alpha \times id) \\
&= ev \circ (id \times ev) \circ (id \times (id \times ev)) \circ \beta.
\end{aligned}
$$

The equation $(id \times \alpha) \circ \alpha \circ (\alpha \times id) = \beta$ holds due to the commutativity of the pentagon diagram of the associator α.

□

Chapter 3

Category of categories

There exists no category of categories by the Russel's paradox. Still we recognize that the structure of categories has a categorical flavour to it. Here we use the concept of meta categories, that the Grothendieck universes permit, to simulate a 'category' that contains all categories. Since categories are by definition restricted to the Grothendieck universe U, we may define a U^+-category that consists of all U-categories. Hence we evade the Russel's paradox.

3.1 Meta category of categories

We apply the general theory of categories into one specific category, namely, the meta-category of categories **M-CAT**. Because our work is contextualized in an arbitrary Grothendieck universe U and the meta category of categories lives in the universe U^+, we obtain the previous definitions and results to the universe U^+. We will see that products, coproducts and exponentials exist in category of small categories. These exponentials are called functor categories. Our task will be to find the morhpisms of the functor categories. The morphisms will be called natural transformations.

Definition 3.1 (Product of categories)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We define $\mathbf{C} \times \mathbf{D}$ to be the category where the

1. objects are pairs (c, d) where c and d are objects in categories $\mathbf{C}$ and $\mathbf{D}$, respectively,
2. morphisms are pairs $(f, g) : (c, d) \to (c', d')$ where $f : c \to c'$ and $g : d \to d'$ in the respective categories and
3. composition is defined componentwise,

$$\begin{tikzcd} (c,d) \arrow[r, "(f,g)"'] \arrow[rr, bend left, "(f',g')\circ(f,g)=(f'f,g'g)"] & (c',d') \arrow[r, "(f',g')"'] & (c'',d''). \end{tikzcd}$$

There exists projections $\pi_1 : \mathbf{C} \times \mathbf{D} \to \mathbf{C}$ and $\pi_2 : \mathbf{C} \times \mathbf{D} \to \mathbf{D}$. They are functorial and the tuple $(\mathbf{C} \times \mathbf{D}, \pi_1, \pi_2)$ satisfies the universal property or product object.

In similar fashion, we define the coproduct of categories by taking the disjoint union of objects and morphisms. The canonical inclusions will be the injections of coproduct.

3.1.1 Subcategories

To study the meta category of categories, we may apply the same concepts that were looked at in the previous study of arbitrary categories, namely monomorphisms, epimorphisms, subobjects and quotient objects.

Definition 3.2. Let $\mathbf{C}$ be a category. Let $\mathcal{A}$ consist of objects A_0 and morphisms A_1 of $\mathbf{C}$. Then $\mathcal{A}$ is called a subcategory of $\mathbf{C}$ if A_1 is closed under composition, the identities of objects in A_0 belong the set of morphisms A_1 and the domains and codomains of morphisms in A_1 exists in A_0. If the inclusion functor $\mathcal{A} \hookrightarrow \mathbf{C}$ is full, we say that $\mathcal{A}$ is a full subcategory of $\mathbf{C}$.

Example 3.3.

1. The category **Mon** of monoids has a full subcategory **Grp** of groups, which again has a full subcategory **Ab** of abelian groups.

2. Let $\mathbf{C}$ be a category. Any collection of morphisms $M \subset \mathrm{Mor}(\mathbf{C})$ defines the smallest subcategory $\mathcal{A}$ of $\mathbf{C}$ that contains the morphisms M via The Fundamental Theorem of Recursion 1.2. This subcategory is said to be the subcategory generated by M. The category $\mathrm{Mor}(\mathcal{A})$ consists of all composites of non-empty finite composable sequences with the addition of right and left identities of the morphisms in M by Corollary 1.12.

3. Any collection O of objects of $\mathbf{C}$ defines a full subcategory of $\mathbf{C}$ and a minimal subcategory $\mathbf{C}$ where the set of objects is O. Minimal is attained by only adding the identities and the full is attained by taking all possible morphisms between the objects in O.

4. Every category $\mathbf{C}$ has a maximal groupoid subcategory defined through the collection of isomorphisms.

Theorem 3.4. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. Then the following hold:*

1. *The functor F is monic if and only if F is injective on morphisms (hence on objects). Additionally, full monic functors are embeddings with respect to the forgetful functor to the morphisms.*

2. *The subobjects of $\boldsymbol{C}$ correspond canonically with the subcategories.*

3. *The categorical image of F, denoted $\boldsymbol{Im}(F)$, is the smallest subcategory of $\boldsymbol{D}$ containing the set theoretical image of F.*

Proof.

1. If F is injective on morphisms, we see that F is injective on objects, because objects correspond to identities. It follows that F is monic. For the converse assume that F is monic and $F(f) = F(g)$ for morphisms f and g in $\mathbf{C}$. Denote the diagrams of f and g by $\overline{f} : \mathbf{2} \to \mathbf{C}$ and $\overline{g} : \mathbf{2} \to \mathbf{C}$. Now $F \circ \overline{f} = F \circ \overline{g}$ and thus $\overline{f} = \overline{g}$ and so $f = g$.

   Since functor between small categories can be thought as L-model morphisms where L is the vocabulary categories. Since full monic functors are globally full injections, we attain that full monic functors are embeddings.

2. Since the category **Cat** can be identified with some category $\mathbf{Model}_L^T$ and all monics of **Cat** are injective, then by the Example 2.45(3) the canonical association of subcategories of $\mathbf{C}$ to subobjects of $\mathbf{C}$ is bijective.

3. We get that $F = i \circ F' : \mathbf{C} \to \mathbf{I} \to \mathbf{D}$, where i is the inclusion and F' the corestriction of F. Given any other such factorization $F = jF''$, we may assume that F'' is also a corestriction and j an inclusion, it follows that $\mathbf{I}$ is contained in the domain of j and thus it follows that $i \leq j$. Hence the categorical image of F is the smallest subcategory of the domain that contains the set theoretical image of F.

□

3.1.2 Quotient categories

In this chapter we follow the paper "Generalized congruences" [2] by Bednarczyk, Borzyszkowski and Pawlowski. Here we characterize extremal epimorphisms and show that they define a strict subclass of epimorphisms.

Consider the embedding of one object categories $\mathbb{N} \hookrightarrow \mathbb{Z}$. This functor is an epimorphism with categorical image not equal to $\mathbb{Z}$. Thus not all epimorphisms are extremal epimorphisms in the category of small categories.

Definition 3.5. (Kernel of a functor) Let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We may extend the function $F : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{D})$ to be a partial function, denoted $F^+ : \mathrm{Mor}(C)^+ \rightharpoondown \mathrm{Mor}(\mathbf{D})$ from the set $\mathrm{Mor}(\mathbf{C})^+$ of non-empty finite sequences of morphisms, where $(f_n, ..., f_1) \mapsto F(f_n) \ldots F(f_1)$

whenever the latter is defined. We denote the domain of F^+ by M_F^+. Since F^+ is a partial function, the function F^+ defines a partial equivalence (not necessarily reflexive) relation $\sim_F$ on the set $\mathrm{Mor}(\mathbf{C})^+$, where two sequences are identified, if and only if F^+ maps them to the same element. We call the relation $\sim_F$ the kernel of the functor F.

We immediately see that $\phi = (f_n, \dots, f_1) \in M_F^+$ if and only if $\mathrm{dom}(F(f_{i+1})) = \mathrm{cod}(F(f_i))$ for all $i < n$. Additionally, $\phi \in M_F^+$ is equivalent with $\phi \sim_F \phi$.

Theorem 3.6 (Functor Colift Theorem)**.** *Assume that the diagram*

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, "F"] \arrow[d, "G"'] & \boldsymbol{E} \\ \boldsymbol{D} & \end{tikzcd} \tag{3.1}$$

is of functors and the categorical image $\boldsymbol{Im}(G)$ *equals* $\boldsymbol{D}$*. Then there exists a unique functor* $I : \boldsymbol{D} \to \boldsymbol{E}$ *that makes the diagram*

$$\begin{tikzcd} \boldsymbol{C} \arrow[r, "F"] \arrow[d, "G"'] & \boldsymbol{E} \\ \boldsymbol{D} \arrow[ur, "I"'] & \end{tikzcd}$$

commute if and only if $\sim_G \subset \sim_F$*. Additionally,* I *is monic if and only if* $\sim_G = \sim_F$*.*

Proof. Notice that $\sim_G \subset \sim_F$ implies that $M_G^+ \subset M_F^+$, since assuming the former and $\phi \in M_G^+$, it follows that $\phi \sim_G \phi$ and hence $\phi \sim_F \phi$. Thus $\phi \in M_F^+$.

Assume that F factors through G with I. Now assume that $(f_n, \dots, f_1) = \phi \sim_G \psi = (g_m, \dots, g_1)$. Hence the following is well-defined

$$G(f_n) \dots G(f_1) = G(g_m) \dots G(g_1)$$

and thus by applying the functor I on the equality yields that F^+ maps the sequences ψ and ϕ equally and especially it's defined. Thus $\sim_G \subset \sim_F$.

Assume next that $\sim_G \subset \sim_F$. The diagram (3.1) turns to

$$\begin{tikzcd} M_G^+ \arrow[r, "F^+|"] \arrow[d, "G^+"'] & \mathrm{Mor}(\mathbf{E}) \\ \mathrm{Mor}(\mathbf{D}) & \end{tikzcd}$$

Since the categorical image of G is the whole category $\mathbf{D}$ it follows that G^+ becomes a surjective map. Hence by Lemma 1.44 and by the assumption that $\sim_G \subset \sim_F$, we attain that there exists a unique map I that makes the diagram

$$\begin{tikzcd} M_G^+ \arrow[r, "F^+|"] \arrow[d, "G^+"'] & \mathrm{Mor}(\mathbf{E}) \\ \mathrm{Mor}(\mathbf{D}) \arrow[ur, "I"'] & \end{tikzcd}$$

commute. We see that I maps the identities to identities. Let g and g' be morphisms in $\mathbf{D}$, where gg' is defined. Now $g = G^+(\phi)$ and $g' = G^+(\psi)$, where $\phi = (f_m, \dots, f_1)$ and $\psi = (h_n, \dots, h_1)$ for some morphisms $f_1, \dots, h_m$ in $\mathbf{C}$. Since gg' is defined, $(f_m, \dots, f_1, h_n, \dots, h_1) \in M_G^+$. Thus

$$\begin{aligned} I(gg') &= I(G^+(f_m, \dots, h_1)) \\ &= F^+(f_m, \dots, h_1) \\ &= F^+(f_m, \dots, f_1) F^+(h_n, \dots, h_1) \\ &= I(g)I(g'). \end{aligned}$$

Now we have seen that I becomes a functor.

Lastly, we show that I is monic if and only if $\sim_F = \sim_G$. By Lemma 1.44 it follows that if the relations $\sim_G$ and $\sim_F$ are equal, then we have the functor I to be injective on morphisms, i.e.

monic. Assume that I is monic. To use Lemma 1.44, we need to see that the domains M_F^+ and M_G^+ match. So assume that $\phi \in M_F^+$. Now $\phi = (f_n, \dots, f_1)$ for some morphisms $f_1, \dots, f_n$ in $\mathbf{C}$. Now we have the following:

$$F(f_n)\dots F(f_1) = I(G(f_n))\dots I(G(f_1)).$$

Since I is injective on morphisms the composition $G(f_n)\dots G(f_1)$ is defined. Thus $\phi \in M_G^+$. Therefore $M_F^+ = M_G^+$ and so the set theoretical kernels match. In other words $\sim_F = \sim_G$. □

Let $\mathbf{C}$ be a category. We define the domain and codomain of a finite non-empty sequence $(f_n, \dots, f_1)$ of morphisms in $\mathbf{C}$ to be $\mathrm{dom}(f_1)$ and $\mathrm{cod}(f_n)$, respectively. Any functor leaving $\mathbf{C}$ defines a partial equivalence relation $\sim$ on the set $\mathrm{Mor}(\mathbf{C})^+$. Motivated by the functor kernel we give the following definition for a generalized congruence:

Definition 3.7 (Generalized congruence)**.** Let $\mathbf{C}$ be a category. Let $\sim$ be partial equivalence relation on the set $\mathrm{Mor}(\mathbf{C})^+$ of finite non-empty sequences of morphisms in $\mathbf{C}$. We denote the induced relation on objects also with $\sim$. The partial equivalence relation $\sim$ is called a generalized congruence on the category $\mathbf{C}$ if the following conditions hold for all $\alpha, \beta, \gamma \in \mathrm{Mor}(\mathbf{C})^+$:

1. If for the concatenated sequence $\alpha\beta$ holds that $\alpha\beta \sim \gamma$, then $\mathrm{dom}(\alpha) \sim \mathrm{cod}(\beta)$.
2. If $\alpha \sim \beta$, then $\mathrm{dom}(\alpha) \sim \mathrm{dom}(\beta)$ and $\mathrm{cod}(\alpha) \sim \mathrm{cod}(\beta)$.
3. If $\alpha \sim \beta$, $\gamma \sim \eta$ and $\mathrm{dom}(\alpha) \sim \mathrm{cod}(\gamma)$, then $\alpha\gamma \sim \beta\eta$.
4. If g and f are composable morphisms in $\mathbf{C}$ then $(gf) \sim (g, f)$.

Notice that the functor kernel is always a partial equivalence relation that satisfies the conditions above. Let $\mathbf{C}$ be a category and $\sim$ a partial equivalence on $\mathbf{C}$. Now the relation $\sim$ defines an equivalence relation on

$$M = \{\alpha \in \mathrm{Mor}(C)^+ \mid \alpha \sim \alpha\}.$$

The set M contains sequences of length one due to the condition four, symmetry and transitivity of relation $\sim$. Therefore the induced relation on objects is an equivalence relation also. Since the definition of a generalized congruence consists of closure conditions, by the Fundamental Theorem of Recursion, any relation R on $\mathrm{Mor}(\mathbf{C})^+$ generates the smallest generalized congruence on $\mathrm{Mor}(\mathbf{C})^+$ that contains R.

Next we will show that a generalized congruence defines canonically a category.

Theorem 3.8 (Quotient category)**.** *Let $\boldsymbol{C}$ be a category with a generalized congruence $\sim$. Denote $M = \{\alpha \in Mor(\boldsymbol{C}) \mid \alpha \sim \alpha\}$. We define*

$$Q = Q_\sim : Mor(\boldsymbol{C}) \hookrightarrow M \to M/\sim, f \mapsto [(f)].$$

Then there exists a unique categorical structure on $M/\sim$, denoted $\boldsymbol{C}/\sim$, such that the objects are the equivalence classes of the objects of $\boldsymbol{C}$ and Q becomes a functor where $\boldsymbol{Im}(Q) = \boldsymbol{C}/\sim$ and $\sim_Q = \sim$.

Proof. The uniqueness follows from Theorem 3.6, because if two structures were given, there would be a unique isomorphism I between them where $IQ = Q$. Since Q has a full image and so every morphism in $M/\sim$ is a finite composition of morphisms of the form $Q(f)$, I must be the identity functor. Hence the composition on $M/\sim$ is uniquely defined.

It remains to define the structure $\mathbf{C}/\sim_F$:

1. Objects are equivalence classes of objects in $\mathbf{C}$ where a pair of objects are identified if and only if their identities are identified by $\sim$.
2. The set of morphisms is $M/\sim$. Let $\alpha \in M$. We define

$$\mathrm{dom}([\alpha]) = [\mathrm{dom}(\alpha)] \text{ and } \mathrm{cod}([\alpha]) = [\mathrm{cod}(\alpha)].$$

3. If $[a] \xrightarrow{[\phi]} [b] \xrightarrow{[\psi]} [c]$ we define $[\psi][\phi] = [\psi\phi]$.

We need to check that $\mathbf{C}/\sim_F$ becomes a category, Q becomes a functor with full image and $\sim_Q=\sim$. Firstly we recognize that the domain and codomain functions are well-defined by the second condition of generalized congruence. Secondly we see that composition is well-defined by the third condition of the definition of generalized congruence. Associativity of the composition is directly seen from the definition of composition. The identity of object $[a]$ is $[id_a]$. Hence $\mathbf{C}/\sim$ is a category.

Now we see that Q becomes functor with a full image and that the functor kernel Q matches $\sim$. Since Q takes morphisms to their equivalence classes, it must map objects also to their equivalence classes. Thus Q is a well-defined function on objects and morphisms. By definition Q maps identities to identities and using the fourth condition of generalized congruences, Q maps compositions to the corresponding compositions. Thus Q is a functor. By the first condition every morphism in $\mathbf{C}/\sim$ is a finite composition of morphism in the set theoretical image of Q. Hence Q has a full image. By the definition of composition we have the equality $\sim=\sim_Q$, since the equation $[(f_n,\dots,f_1)] = [(f_n)]\dots[(f_1)]$ holds for all $(f_n,\dots,f_1)\in M$. ☐

Remark 3.9. We have generalized The Fundamental Theorem of Homomorphism to categories: Let $F:\mathbf{C}\to\mathbf{D}$ be a functor. Then the induced functor defines an isomorphism $\mathbf{C}/\sim_F\cong \mathbf{Im}(F)$.

3.2 Exponentiation of categories

Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We would like to define the exponential category $\mathbf{D}^{\mathbf{C}}$. We know that if such a category exists, the objects must match with functors $F:\mathbf{C}\to\mathbf{D}$.

Definition 3.10. We call a functor F, whose domain is a product of two categories, a bifunctor.

Theorem 3.11 (Bifunctor Decomposition Theorem). *Let $F:\boldsymbol{C}\times\boldsymbol{D}\to\boldsymbol{E}$ be a bifunctor. The following definitions yield functors:*

$$F_c:\boldsymbol{D}\to\boldsymbol{E}, d\mapsto F(c,d), g\mapsto F(id_c,g)$$

$$F^d:\boldsymbol{C}\to\boldsymbol{E}, c\mapsto F(c,d), f\mapsto F(f,id_d)$$

Additionally,

$$F(f)F(g)=F(f,g)=F(g)F(f)$$

holds for all morphisms f and g in $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively.[1]

The converse is true in the following sense: Consider two collections of functors $(G_c:\boldsymbol{D}\to\boldsymbol{E})_c$ and $(G^d:\boldsymbol{C}\to\boldsymbol{E})_d$ where the indices c and d are over the class of objects of $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively. Assume the compatibility conditions:

$$G_c(d)=G^d(c) \text{ for objects } c \text{ and } d \text{ in } \boldsymbol{C} \text{ and } \boldsymbol{D}\text{, respectively, and}$$

$$G^{d'}(f)G_c(g)=G_{c'}(g)G^d(f) \text{ for morphisms } c\xrightarrow{f}c' \text{ and } d\xrightarrow{g}d' \text{ in } \boldsymbol{C} \text{ and } \boldsymbol{D}\text{, respectively.}$$

Then there exists a unique functor

$$G:\boldsymbol{C}\times\boldsymbol{D}\to\boldsymbol{E}$$

where $G(c,d)=G_c(d)$ and $G(f,g)=G^{d'}(f)G_c(g)$ for all objects c,c',d,d' and morphisms $c\xrightarrow{f}c'$ and $d\xrightarrow{g}d'$ in their respective categories $\boldsymbol{C}$ and $\boldsymbol{D}$.

Proof. First we are going to see that fixing c in $\mathbf{C}$ yields a functor $F_c:\mathbf{D}\to\mathbf{E}$. Let d be an object in $\mathbf{D}$. Now $F_c(id_d)=F(id_c,id_d)=id_{F(c,d)}$. Let $d\xrightarrow{g}d'\xrightarrow{g'}d''$ be morphisms in $\mathbf{D}$. Here

$$\begin{aligned}F_c(g'g)&=F(id_c,g'g)\\&=F((id_c,g')\circ(id_c,g))\\&=F(id_c,g')F(id_c,g)\\&=F_c(g')F_c(g).\end{aligned}$$

[1] $F(f)$ and $F(g)$ are shorthand to $F^{d'}(f), F^d(f), F_{c'}(g)$ and $F_c(g)$ where $c\xrightarrow{f}c'$ and $d\xrightarrow{g}d'$ such that the notation is defined. The lack of inserting the objects to the notation is reasonable, since the insertion is unique.

Therefore F_c is a functor. Similarly F^d is a functor for all objects d in $\mathbf{D}$. Next we will show that given $g : d \to d'$ and $f : c \to c'$,

$$F^{d'}(f)F_c(g) = F(f, g) = F_{c'}(g)F^d(f).$$

This holds, since

$$\begin{aligned}
F^{d'}(f)F_c(g) &= F(f, id_{d'})F(id_c, g)\\
&= F((f, id_{d'}) \circ (id_c, g))\\
&= F(f, g)\\
&= F((id_{c'}, g) \circ (f, id_d))\\
&= F(id_{c'}, g)F(f, id_d)\\
&= F_{c'}(g)F^d(f).
\end{aligned}$$

Next we will show the converse. Assume that we have the collections of functors $(G_c)_c$ and $(G^d)_d$ as in the statement. Define

$$\begin{aligned}
G : \mathbf{C} \times \mathbf{D} \to \mathbf{E}, (c, d) &\mapsto G_c(d),\\
((c, d) \xrightarrow{(f,g)} (c', d')) &\mapsto G^{d'}(f)G_c(g).
\end{aligned}$$

We see that G is well-defined as a function on objects. Given a morphism $(f, g) : (c, d) \to (c', d')$ in $\mathbf{C} \times \mathbf{D}$, we obtain $G_c(g) : G_c(d) \to G_c(d')$ and $G^{d'}(f) : G^{d'}(c) \to G^{d'}(c')$. By assumption, $G_c(d') = G^{d'}(c)$, it follows that the composition $G^{d'}(f)G_c(g)$ is well-defined. Similarly for the other composition. Hence $G(f, g) : G(c, d) \to G(c', d')$ and G is well-defined function on morphisms.

Next we will show that G is a functor. Let (c, d) be an object in $\mathbf{C} \times \mathbf{D}$. Now

$$G(id_c, id_d) = G^d(id_c)G_c(id_d) = id_{G^d(c)} \circ id_{G_c(d)} = id_{G(c,d)}$$

and so G takes identities to identities. Assume $(c, d) \xrightarrow{(f,g)} (c', d') \xrightarrow{(f',g')} (c'', d'')$ are morphisms in $\mathbf{C} \times \mathbf{D}$. Now

$$\begin{aligned}
G((f', g')(f, g)) &= G(f'f, g'g)\\
&= G^{d''}(f'f)G_c(g'g)\\
&= G^{d''}(f')G^{d''}(f)G_c(g')G_c(g)\\
&= G^{d''}(f')G^{c'}(g')G'_d(f)G_c(g)\\
&= G(f', g')G(f, g).
\end{aligned}$$

Therefore G is a functor. The uniqueness of G is seen by the fact that G_c and G^d can be constructed back form G by the previous part of this theorem. $\square$

From the previous proof we see that the constructions $F \mapsto ((F_c)_c, (F^d)_d)$ and $((G_c)_c, (G^d)_d) \mapsto G$ are each others inverses. Importantly, we notice that this process loses no information. Hence we have an equivalent way of looking at bifunctors as functors from a product category and as two collections of functors that satisfy two compatibility conditions.

The equations $F(f)F(g) = F(f, g) = F(g)F(f)$ conveys the information that the following square commutes:

$$\begin{tikzcd}
(c,d) \arrow[d, "{(f,g)}"] \\
(c',d')
\end{tikzcd}
\qquad
\begin{tikzcd}
F(c,d) \arrow[r, "F(g)"] \arrow[d, "F(f)"] \arrow[dr, "{F(f,g)}"] & F(c,d') \arrow[d, "F(f)"] \\
F(c',d) \arrow[r, "F(g)"'] & F(c',d')
\end{tikzcd}$$

We have a corollary relating to all locally small categories and their hom-functors.

Corollary 3.12. *Let $\boldsymbol{C}$ be a locally small category. Then there exists a functor $\boldsymbol{C} : \boldsymbol{C}^{op} \times \boldsymbol{C} \to \boldsymbol{Set}$, where $(a, b) \mapsto \boldsymbol{C}(a, b)$, $\boldsymbol{C}(f^{op}, g)(T) = gTf$ for $f : a' \to a, g : b \to b'$ and $T \in \boldsymbol{C}(a, b)$.*

Proof. Since $\mathbf{C}_a(b) = \mathbf{C}^b(a)$ and

$$\mathbf{C}^{b'}(f^{op})\mathbf{C}_a(g)(T) = gTf = \mathbf{C}_{a'}(g)\mathbf{C}^b(f^{op})(T),$$

we attain the functor $\mathbf{C} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ by applying the Bifunctor Decomposition Theorem. □

Let $\mathbf{C}$ and $\mathbf{D}$ be small categories. Assume for a moment that the exponential category $\mathbf{D}^{\mathbf{C}}$ exists. On one hand every morphism of $\mathbf{D}^{\mathbf{C}}$ corresponds to a diagram $\mathbf{2} \to \mathbf{D}^{\mathbf{C}}$, which on the other hand corresponds to a diagram $F : \mathbf{2} \times \mathbf{C} \to \mathbf{D}$. Denote the unique non-trivial morphism in $\mathbf{2}$ by $!$. Let $F : \mathbf{2} \times \mathbf{C} \to \mathbf{D}$ be a functor. Now F contains the following information by the Bifunctor Decomposition Theorem: The functor F defines functors $F_0, F_1 : \mathbf{C} \to \mathbf{D}$ where the diagram

$$\begin{tikzcd} c \arrow[d, "f"] & F(0,c) \arrow[r, "F(!,id_c)"] \arrow[d, "F_0(f)"] & F(1,c) \arrow[d, "F_1(f)"] \\ d & F(0,d) \arrow[r, "F(!,id_d)"] & F(1,d) \end{tikzcd} \tag{3.2}$$

commutes for all morphisms $f : c \to d$ in $\mathbf{C}$. If there exists morphism $\eta : F_1 \to F_2$ in the exponential category $\mathbf{D}^{\mathbf{C}}$, then η must contain the exact information of that the diagram (3.2) always commutes for any given morphism $f : c \to d$. Hence we must define $\eta : F_1 \to F_2$, $\eta = (\eta_c = F(!, id_c) : F_1(c) \to F_2(c))_c$, if we want η to create the functor F back. For this reason we give the following definition for a morphism between functors.

Definition 3.13 (Natural transformation)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F, G : \mathbf{C} \to \mathbf{D}$ be functors. We call a collection $\eta = (\eta_c : Fc \to Gc)_{c \in \mathrm{Mor}(\mathbf{C})}$ of morphisms in $\mathbf{D}$ associated to functors F and G a natural transformation from F to G, if for any morphisms $f : c \to d$ in $\mathbf{C}$ the diagram

$$\begin{tikzcd} c \arrow[d, "f"] & F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ d & F(d) \arrow[r, "\eta_d"'] & G(d) \end{tikzcd}$$

commutes. We denote then $\eta : F \Rightarrow G : \mathbf{C} \to \mathbf{D}$.

Definition 3.14. (Exponential category) Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Define a meta category $[\mathbf{C}, \mathbf{D}]$, which is also denoted $\mathbf{D}^{\mathbf{C}}$, as follows:

1. The objects are functors $F : \mathbf{C} \to \mathbf{D}$.
2. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors, the morphisms $F \to G$ are the natural transformations $F \Rightarrow G : \mathbf{C} \to \mathbf{D}$.
3. The composition of natural transformations $F \overset{\eta}{\Rightarrow} G \overset{\theta}{\Rightarrow} H : \mathbf{C} \to \mathbf{D}$ is defined by the componentwise composition. The composite natural transformation is denoted by
$$\theta \bullet \eta : F \Rightarrow H : \mathbf{C} \to \mathbf{D}.$$

The composition of natural transformations is well-defined. Every functor has the identity natural transformation as a collection of identities and the composition is clearly associative. Therefore $[\mathbf{C}, \mathbf{D}]$ is a meta category.

Remark 3.15. It is set theoretically important to notice the following: Assume that $\mathbf{C}$ is a small category. If $\mathbf{D}$ is a (locally) small category, then $[\mathbf{C}, \mathbf{D}]$ is also a (locally) small category. This follows directly from the Characterization Theorem 1.66.

Example 3.16.

1. If $\mathbf{C}$ is a category, then $\mathbf{C} \cong \mathbf{C}^{\mathbf{1}}$, and hence morphisms can be thought as natural transformations. Any collection of morphisms $(f_i)_{i \in I}$, where I is an index set, can be thought as a natural transformation between functors $I \rightrightarrows \mathbf{C}$ where one functor chooses the domains of morphisms $f_i, i \in I$ and the other chooses the codomains of $f_i, i \in I$.

2. The arrow category $\mathrm{Ar}(\mathbf{C})$ of a category $\mathbf{C}$ is isomorphic to $\mathbf{C}^{1\to 2}$:

$$\begin{tikzcd} a \arrow[r, "\eta_1"] \arrow[d, mapsto, "f"] & b \arrow[d, mapsto, "g"] \\ a' \arrow[r, "\eta_2"'] & b' \end{tikzcd}$$

3. The category **Quiv** of quivers is isomorphic to $\mathbf{Set}^{1\rightrightarrows 2}$:

$$\begin{tikzcd} E_1 \arrow[rr, bend left, "\eta=(\eta_1,\eta_2)"] \arrow[d, bend right, "s_1"'] \arrow[d, bend left, "t_1"] & & E_2 \arrow[d, bend right, "s_2"'] \arrow[d, bend left, "t_2"] \\ V_1 & & V_2 \end{tikzcd} \qquad \begin{tikzcd} E_1 \arrow[r, "\eta_1"] \arrow[d, mapsto, "s_1"] & E_2 \arrow[d, mapsto, "s_1"] \\ V_1 \arrow[r, "\eta_2"'] & V_2 \end{tikzcd} \qquad \begin{tikzcd} E_1 \arrow[r, "\eta_1"] \arrow[d, mapsto, "t_1"] & E_2 \arrow[d, mapsto, "t_2"] \\ E_1 \arrow[r, "\eta_2"] & E_2 \end{tikzcd}$$

4. If M is a monoid, then the equivariant maps between monoid actions are precisely the same as natural transformations between the functors corresponding the actions.

$$\begin{tikzcd} \mathbf{M} \arrow[r, bend left, "\eta"] \arrow[d, mapsto, "X"] & \mathbf{M} \arrow[d, mapsto, "Y"] \\ \mathbf{Set} & \mathbf{Set} \end{tikzcd} \qquad \begin{tikzcd} * \arrow[d, mapsto, "\forall m\in M"] \\ * \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, mapsto, "X(m)"] & Y \arrow[d, mapsto, "Y(m)"] \\ X \arrow[r, "\eta"'] & Y \end{tikzcd}$$

5. Let $\mathbf{P}$ and $\mathbf{Q}$ be preorder categories. The functors from $\mathbf{P}$ to $\mathbf{Q}$ correspond to increasing maps between the prosets and a natural transformations between such functors contains exactly the information that one map is bounded from above by the other pointwise. Thus the interpretations of the exponential object $\mathbf{Q}^{\mathbf{P}}$ agree as a proset category and as a proset.

Given categories $\mathbf{C}$ and $\mathbf{D}$, we define a functor called the evaluation functor

$$\begin{aligned} Ev : [\mathbf{C},\mathbf{D}] \times \mathbf{C} \to \mathbf{D}, (F, c) &\mapsto Fc \\ (\eta : F_1 \Rightarrow F_2, f) &\mapsto F_2(f)\eta_{\mathrm{dom}(f)}. \end{aligned}$$

Theorem 3.17. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be small categories. Then the evaluation functor Ev, as above defined, is a functor and $(\boldsymbol{D}^{\boldsymbol{C}}, Ev)$ defines an exponential object of the pair $(\boldsymbol{C},\boldsymbol{D})$ in **Cat** of small categories.*

Proof. First we will show that Ev is a functor. Let (F, c) be an object in $[\mathbf{C},\mathbf{D}] \times \mathbf{C}$. Now

$$Ev(I_F, id_c) = F(id_c)id_{Fc} = id_{F_c}.$$

Assume then that $F \overset{\eta}{\Rightarrow} G \overset{\theta}{\Rightarrow} H$ and $c \xrightarrow{f} c' \xrightarrow{g} c''$. Now

$$\begin{aligned} Ev((\theta, g)(\eta, f)) &= Ev(\theta\eta, gf) \\ &= H(gf)(\theta\eta)_c \\ &= H(g)H(f)\theta_c\eta_c \\ &= H(g)\theta_{c'}G(f)\eta_c \\ &= Ev(\theta, g)Ev(\eta, f). \end{aligned}$$

Let $\mathbf{A}$ be a small category. Let $F : \mathbf{A} \times \mathbf{C} \to \mathbf{D}$. We need to show that there exists a unique functor $G : \mathbf{A} \to [\mathbf{C},\mathbf{D}]$ such that the diagram

$$\begin{tikzcd} {[\mathbf{C},\mathbf{D}]} \\ \mathbf{A} \arrow[u, dotted, "\exists! G"] \end{tikzcd} \qquad \begin{tikzcd} {[\mathbf{C},\mathbf{D}] \times \mathbf{C}} \arrow[r, "Ev"] & \mathbf{D} \\ \mathbf{A} \times \mathbf{C} \arrow[u, dotted, "G\times I_{\mathbf{C}}"] \arrow[ur, "F"'] \end{tikzcd} \tag{3.3}$$

commutes. First we will show the uniqueness and then the existence:

Uniqueness: Assume that such a functor $G : \mathbf{A} \to [\mathbf{C}, \mathbf{D}]$ exists. We are going to use the functor decomposition Theorem 3.11 to the collections of functors $(G(a))_a$ and $(G(-)_c)_c$ to attain the functor F, where

$$G(-)_c : \mathbf{A} \to \mathbf{D}, a \mapsto G(a)(c), \alpha \mapsto G(\alpha)_c.$$

To use the Bifunctor Decomposition Theorem we need to show that the compatibility conditions hold and $G(-)_c$ is a functor. It is clear that the functor candidates $G(-)_c$ map identities to identities. Assume that $a \xrightarrow{\alpha} a' \xrightarrow{\beta} a''$ are morphisms in $\mathbf{A}$. Now

$$G(-)_c(\beta\alpha) = G(\beta\alpha)_c = (G(\beta)G(\alpha))_c = G(\beta)_c G(\alpha)_c = G(-)_c(\beta)G(-)_c(\alpha).$$

Thus $G(-)_c$ is a functor for every object c in $\mathbf{C}$. Clearly the compatibility conditions hold for the collections of functors $(G(a))_a, (G(-)_c)_c$. It is left to see that $F(a, c) = G(a)(c)$ and $F(\alpha, f) = G(\alpha)_{c'} G(a)(f)$ for $\alpha : a \to a'$ in $\mathbf{A}$ and $f : c \to c'$ in $\mathbf{C}$. Since the diagram (3.3) commutes by assumption we have for objects a and c in $\mathbf{A}$ and $\mathbf{C}$, respectively

$$\begin{aligned} F(a, c) &= Ev \circ (G \times I_C)(a, c) \\ &= Ev(G(a), c) \\ &= G(a)(c). \end{aligned}$$

Let $\alpha : a \to a'$ and $f : c \to c'$ be morphisms in $\mathbf{A}$ and $\mathbf{C}$, respectively. Now it follows that $G(\alpha) : G(a) \Rightarrow G(a') : \mathbf{C} \to \mathbf{D}$ and

$$\begin{aligned} F(\alpha, f) &= Ev \circ (G \times I_C)(\alpha, f) \\ &= Ev(G(\alpha), f) \\ &= G(a')(f)G(\alpha)_c \\ &= G(\alpha)_{c'} G(a)(f). \end{aligned}$$

Therefore, $F_a = G(a)$ and $F^c = G(-)_c$ for all suitable objects a, c. Thus the functor G is uniquely defined on objects. Given any morphism $\alpha : a \to a'$ in $\mathbf{A}$ we get that $G(\alpha)_c = F^c(\alpha)$. Thus G is fixed on morphisms too. Therefore G is uniquely defined.

Existence: Define a functor $G : \mathbf{A} \to [\mathbf{C}, \mathbf{D}]$ by setting

$$\begin{aligned} a &\mapsto F_a \\ (\alpha : a \to a') &\mapsto (F^c(\alpha) : F_a(c) \to F_{a'}(c))_c \end{aligned}$$

We need to check that G is well-defined and a functor. Let $\alpha : a \to a'$ be a morphism in $\mathbf{A}$. For any given morphism $f : c \to c'$ in $\mathbf{C}$ we need to see that the diagram

$$\begin{tikzcd} c \arrow[d, "f"] & F_a(c) \arrow[r, "G(\alpha)_c"] \arrow[d, "F_a(f)"] & F_{a'}(c) \arrow[d, "F_{a'}(f)"] \\ c' & F_a(c') \arrow[r, "G(\alpha)_{c'}"'] & F_{a'}(c) \end{tikzcd}$$

commutes. Notice that

$$\begin{aligned} F_{a'}(f)G(\alpha)_c &= F(id_{a'}, f)F(\alpha, id_c) \\ &= F(\alpha, f) \\ &= F(\alpha, id_{c'})F(id_a, f) \\ &= G(\alpha)_{c'} F_a(f). \end{aligned}$$

Thus we see that $G(\alpha)$ is a natural transformation and G becomes a functor, since it maps identities to identities and it respects the composition, which follows directly from the fact that $G(\alpha)_c = F^c(\alpha)$.

We immediately notice that $G(a)(c) = F(a, c)$ and $G(a')(f)G(\alpha)_c = F(\alpha, f)$ for objects a, a' in $\mathbf{A}$, c, c' in $\mathbf{C}$ and morphisms $\alpha : a \to a'$ and $f : c \to c'$. Therefore the diagram (3.3) commutes. $\square$

Remark 3.18. We are now able to interchange functors $\mathbf{C} \times \mathbf{D} \to \mathbf{E}$ with functors $\mathbf{C} \to [\mathbf{D}, \mathbf{E}]$ canonically via the exponential tranposition.

Since the category **Cat** is a Cartesian closed category, we have an immediate corollary for the existence of a bifunctor called the horizontal composition of natural transformations.

Corollary 3.19. *Let $\boldsymbol{C}, \boldsymbol{D}, \boldsymbol{E}$ be small categories. Then there exists a unique bifunctor*

$$\begin{aligned} * : [\boldsymbol{D}, \boldsymbol{E}] \times [\boldsymbol{C}, \boldsymbol{D}] \to [\boldsymbol{C}, \boldsymbol{E}], (G, F) \mapsto G \circ F \\ (I_G * \eta)_c = G(\eta_c) \\ (\theta * I_F)_c = \theta_{Fc}, \end{aligned}$$

called the horizontal composition. Additionally, the horizontal composition is associative and it has identity elements as the identity natural transformations on identity functors.

Proof. The uniqueness is seen from the Bifunctor Decomposition Theorem, since the decomposition functors are fixed. From the internal composition Theorem 2.64 we obtain a bifunctor

$$* : [\mathbf{D}, \mathbf{E}] \times [\mathbf{C}, \mathbf{D}] \to [\mathbf{C}, \mathbf{E}]$$

as the unique functor that makes the diagram

$$\begin{tikzcd} {[\mathbf{C}, \mathbf{E}] \times \mathbf{C}} \arrow[rr, "Ev"] & & \mathbf{E} \\ ([\mathbf{D}, \mathbf{E}] \times [\mathbf{C}, \mathbf{D}]) \times \mathbf{C} \arrow[u, dashed, "* \times I_{\mathbf{C}}"] \arrow[r] & {[\mathbf{D}, \mathbf{E}] \times ([\mathbf{C}, \mathbf{D}]) \times \mathbf{C})} \arrow[r, "I \times Ev"'] & {[\mathbf{D}, \mathbf{E}] \times \mathbf{D}} \arrow[u, "Ev"'] \end{tikzcd}$$

commute. So the functor $*$ is attained from the composite functor $T : ([\mathbf{D}, \mathbf{E}] \times [\mathbf{C}, \mathbf{D}]) \times \mathbf{C} \to \mathbf{E}$ of the commuting diagram above. For fixed $G \overset{\theta}{\Rightarrow} G' : \mathbf{D} \to \mathbf{E}$, $F \overset{\eta}{\Rightarrow} F' : \mathbf{C} \to \mathbf{D}$ and an object c in $\mathbf{C}$, we have

$$\begin{aligned} T((G, F), c) &= G(F(c)) \\ &\text{and} \\ T((\theta, \eta), f) &= Ev(\theta, F'(f)\eta_c) \\ &= G'(F'(f)\eta_c)\theta_{F_c} \\ &= G'(F'(f))G'(\eta_c)\theta_{Fc}. \end{aligned}$$

Moreover,

$$\begin{aligned} *(G, F) &= T_{(G,F)} = G \circ F, \\ *(I_G, \eta)_c &= T((I_G, \eta), id_c) = G(\eta_c) \\ &\text{and} \\ *(\theta, I_F)_c &= T((\theta, I_F), id_c) = \theta_{Fc}. \end{aligned}$$

Associativity and the existence of identities of the horizontal composition follows from the Internal Composition Theorem. □

Remark 3.20. Since a functor F always specifies an identity natural transformation $I = id_F$ on it, we can denote $F = id_F$. It is clear from the context, whether we are talking about the identity natural transformation on a functor or the functor itself. Now we know that $F * \eta$ refers to the natural transformation of $I_F * \eta$. Furthermore for natural transformations η and θ, where $\theta * \eta$ is defined, we may denote $\theta\eta := \theta * \eta$.

Theorem 3.21. *Let $G : \boldsymbol{D} \to \boldsymbol{E}$ be a fully faithful functor. Then $G * (-) : [\boldsymbol{C}, \boldsymbol{D}] \to [\boldsymbol{C}, \boldsymbol{E}]$ is also a fully faithful functor, where $\boldsymbol{C}$ is any category.*

Proof. Let the following be a diagrammatic lifting problem:

$$\begin{tikzcd} & {[\mathbf{C}, \mathbf{D}]} \arrow[d, "G*(-)"] \\ \mathbf{A} \arrow[r, "D"'] & {[\mathbf{C}, \mathbf{E}]} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Obj}([\mathbf{C}, \mathbf{D}]) \arrow[d, "G*(-)"] \\ \mathrm{Obj}(\mathbf{A}) \arrow[ur, "D'"] \arrow[r, "D"'] & \mathrm{Obj}([\mathbf{C}, \mathbf{E}]) \end{tikzcd}$$

It suffices to show that this lifting problem has a unique solution. Notice that solutions to the given lifting problem corresponds bijectively with the solutions of the following lifting problem:

$$\begin{tikzcd} & \mathbf{D} \arrow[d, "G"] \\ \mathbf{A}\times\mathbf{C} \arrow[r, "\tilde{D}"'] & \mathbf{E} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{Obj}(\mathbf{D}) \arrow[d, "G"] \\ \mathrm{Obj}(\mathbf{A}\times\mathbf{C}) \arrow[r, "\tilde{D}"'] \arrow[ur, "\tilde{D}'"] & \mathrm{Obj}(\mathbf{E}) \end{tikzcd}$$

Since G is fully faithful the latter lifting problem has exactly one solution and hence the former has exactly one solution. Thus $G*(-)$ is fully faithful. □

3.2.1 Natural transformations

Let's start with a few examples. Natural transformations refer to a canonically constructed morphism between objects. Very often natural transformations arise out of general constructions.

Example 3.22.

1. Let M be an R-module where R is a commutative ring. Then the canonical map from M to its double dual M^{**} becomes a natural transformation. First notice that the hom-functor $\mathrm{Hom} : \mathbf{R\text{-}Mod}^{op} \times \mathbf{R\text{-}Mod} \to \mathbf{Set}$ becomes naturally a left R-module valued functor, since the pointwise definitions of addition and scalar multiplication work well because of the commutative nature of $(M,+)$ and of $(R,\cdot)$. Hence we get $\mathrm{Hom} : \mathbf{R\text{-}Mod}^{op} \times \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$. We define the duality $*$ as the functor $\mathrm{Hom}^R : \mathbf{R\text{-}Mod}^{op} \to \mathbf{R\text{-}Mod}$ and the double dual functor is $D = *(*^{op}) : \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$. Define a function

$$\eta = \eta_M : M \to M^{**}, m \mapsto \tilde{m},$$

where $\tilde{m}(n) = n(m)$ for all $n \in M^*$. We see that $\tilde{m}$ is linear, since

$$\tilde{m}(kn + n') = (kn + n')(m) = kn(m) + n'(m) = k\tilde{m}(n) + \tilde{m}(n')$$

for all $k \in R$ and $n, n' \in M^*$. Thus η is defined as a function. Additionally, η becomes a linear map itself, since

$$\widetilde{km + m'}(n) = n(km + m') = kn(m) + n(m') = k\tilde{m}(n) + \tilde{m}'(n)$$

for all $k \in R$ and $m, m' \in M$. To see that $\eta = (\eta_M)_M$ becomes a natural transformation $I_{\mathbf{R\text{-}Mod}} \Rightarrow D : \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$, we need to show that the following diagram commutes:

$$\begin{tikzcd} \forall M \arrow[d, dashed, "\forall L"] \\ \forall N \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, dashed, "L"] & M^{**} \arrow[d, dashed, "D(L)"] \\ N \arrow[r, "\eta_N"'] & N^{**} \end{tikzcd}$$

Let M, N be R-modules and let $L : M \to N$ be a linear map. We need to show that $D(L)\eta_M = \eta_N L$. Fix $m \in M$. Now

$$\tilde{m} \circ L^*(x) = \tilde{m}(x \circ L) = x \circ L(m) = x(L(m)) = \widetilde{L(m)}(x) = (\eta_N(L(m)))(x) = (\eta_N \circ L(m))(x)$$

for all $x \in M^*$. Hence $\tilde{m} \circ L^* = \eta_N \circ L(m)$. Finally,

$$D(L)\eta_M(m) = L^{**}(\tilde{m}) = \tilde{m} \circ L^* = \eta_N \circ L(m).$$

Therefore η is a natural transformation.

2. Embedding a set to it's power set, where the elements of a set are mapped to the corresponding singleton sets, is a natural map.

3. There are at least four natural transformations $\mathcal{P}^* \Rightarrow \mathcal{P}^* : \mathbf{Set}^{op} \to \mathbf{Set}$. They are the constant maps of empty set and the whole set, identity and complementation. Since in all of those cases the diagrams

$$\begin{tikzcd} \forall X \arrow[d, dashed, "f"] & \mathcal{P}^*(Y) \arrow[r] \arrow[d, "f^{-1}"'] & \mathcal{P}^*(Y) \arrow[d, "f^{-1}"] \\ \forall Y & \mathcal{P}^*(X) \arrow[r] & \mathcal{P}^*(X) \end{tikzcd}$$

commute.

Now we are going to look at one very useful basic way of finding natural transformations that are isomorphisms.

Theorem 3.23. *Let $\eta : F \Rightarrow G : \boldsymbol{C} \to \boldsymbol{D}$ be a natural transformation. Then η is an isomorphism if and only if the morphism η_c is an isomorphism for all objects c in $\boldsymbol{C}$.*

Proof. Components of η are isomorphisms, given that the natural transformation η is. Assume then that η_c is an isomorphism for all c in $\mathbf{C}$. Now define $\tau_c = \eta_c^{-1} : Gc \to Fc$. Its enough to see that given $f : c \to c'$, we get $\tau_{c'} Gf = Ff\tau_c$ which is equivalent with $\eta_{c'} Ff = Gf\eta_c$. Hence τ is a natural transformation and the inverse of η. □

We may ask how well does this isomorphism of functors preserve monic, epic, isomorphic, full, faithful and dense functors. It so happens to be that only fullness, faithfulness and denseness are preserved.

Theorem 3.24. *Let $\eta : F \Rightarrow G : \boldsymbol{C} \to \boldsymbol{D}$ be a natural isomorphism. Then the following conditions hold:*

1. *If F is full (faithful), then G is full (faithful).*

2. *If F is dense, so is G.*

3. *Let $\boldsymbol{P}$ be a proset category, where $a \leq b$ for all $a, b \in \boldsymbol{P}$. Then all endofunctors on $\mathbf{P}$ are isomorphic. Especially, if $\mathbf{P}$ consists of at least two objects, then a constant endofunctor on $\boldsymbol{P}$ is neither epic nor monic, but it is isomorphic to the identity functor.*

Proof.

1. We use Theorem 2.29 which characterized faithfulness and fullness by lifting problems. Fix the category $\mathbf{I} := \bullet \to \bullet$, a functor $D : \mathbf{I} \to \mathbf{D}$ and a function $D' : \mathrm{Obj}(\mathbf{I}) \to \mathrm{obj}(\mathbf{C})$:

$$\begin{tikzcd} & \mathrm{obj}(\mathbf{C}) \arrow[d, dashed, bend right, "F"'] \arrow[d, dashed, bend left, "G"] \arrow[d, phantom, "\cong"] \\ \mathrm{obj}(\mathbf{I}) \arrow[ur, "D'"] \arrow[r, "D"'] & \mathrm{obj}(\mathbf{D}) \end{tikzcd}$$

If F solves the lifting problem defined by the pair (D, D'), then G solves the corresponding lifting problem $(\eta * D, D')$. Similarly for the inverse of η. These operations are each other's inverses and so we have a bijection with the lifting problems. If F is faithful, it has at most one solution to its lifting problems and so does G and G is faithful. If F is full, then F has at least one solution to the lifting problem and so does G. Thus G would be full.

2. This is clear.

3. Since any function $\mathbf{P} \to \mathbf{P}$ is increasing, it becomes a functor. For any two functions $f, g : \mathbf{P} \to \mathbf{P}$ holds that $f \leq g \leq f$ and so $f \cong g$.

□

Theorem 3.25. *Let $\eta : F \Rightarrow G : \boldsymbol{C} \to \boldsymbol{D}$ be a natural isomorphism. If F preserves, reflects or creates initial objects up to isomorphism, then so does G, respectively.*

Proof. Let x be an object in $\mathbf{C}$. Assume that Fx is an initial object. Thus $\mathbf{D}_{Fx}(\eta_d) : \mathbf{D}(Fx, d) \cong \mathbf{D}(Gx, d)$ for every object d in $\mathbf{D}$. Thus Gx is is an initial object. Thus if F preserves initial objects, so does G.

Assume that F reflects initial objects. Let c be an object in $\mathbf{C}$ and assume that Gc is an initial object. Since $Gc \cong Fc$, Fc is an initial object and thus c is an initial object. Assume that F creates initial objects and d is initial in $\mathbf{D}$. Hence there exists c' where $Fc' = d$ and c' is an initial object. Thus $Gc' \cong d$ is an initial object. Furthermore G also reflects initial objects and hence creates initial objects up to an isomorphism. $\square$

For a bifunctor we have seen three different representations as itself, its component functors and as functors attained by exponential transposition. Natural transformations behave nicely in these changed perspectives:

Theorem 3.26. *Let $\boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ be categories and let $B, T : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$ be bifunctors. Suppose η is a collection of morphisms $\eta_{(c,d)} : B(c,d) \to T(c,d)$ for object c, d in $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively. Then the following conditions are equivalent:*

1. *$\eta : B \Rightarrow T : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$.*
2. *$\eta_{(c,-)} : B_c \Rightarrow T_c : \boldsymbol{D} \to \boldsymbol{E}$ and $\eta_{(-,d)} : B^d \Rightarrow T^d : \boldsymbol{C} \to \boldsymbol{E}$, where $\eta_{(c,-)} = (\eta_{(c,d')})_{d'}$ and $\eta_{(-,d)} = (\eta_{(c',d)})_{c'}$ for all objects c, d in their respective categories $\boldsymbol{C}$ and $\boldsymbol{D}$.*
3. *$\tilde{\eta} : \tilde{B} \Rightarrow \tilde{T} : \boldsymbol{C} \to [\boldsymbol{D}, \boldsymbol{E}]$, where $\tilde{\eta} = ((\eta_{(c,-)})_c)$.*

Proof. We will prove the claim in the order $1 \Leftrightarrow 2 \Leftrightarrow 3$

$1 \Leftrightarrow 2$: Assume that $\eta : B \Rightarrow T : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$. Let $f : c \to c'$ and $g : d \to d'$ be a morphism in $\mathbf{C}$ and $\mathbf{D}$, respectively. We need to see that the diagrams

$$\begin{tikzcd} c \arrow[d, mapsto, "f"] & B(c,d) \arrow[r, "\eta_{(c,d)}"] \arrow[d, mapsto, "{B(f,id_d)}"] & T(c,d) \arrow[d, mapsto, "{T(f,id_d)}"] \\ c' & B(c',d) \arrow[r, "\eta_{(c',d)}"'] & T(c',d) \end{tikzcd} \qquad \begin{tikzcd} d \arrow[d, mapsto, "g"] & B(c,d) \arrow[r, "\eta_{(c,d)}"] \arrow[d, mapsto, "{B(id_c,g)}"] & T(c,d) \arrow[d, mapsto, "{T(id_c,g)}"] \\ d' & B(c,d') \arrow[r, "\eta_{(c,d')}"'] & T(c,d') \end{tikzcd}$$

commute, but this is clear. For the converse assume that $\eta_{(c,)}$ and $\eta_{(-,d)}$ are natural transformations for objects c and d. Let $(f,g) : (c,d) \to (c',d')$ be a morphism in $\mathbf{C} \times \mathbf{D}$. We need to see that the diagram

$$\begin{tikzcd} (c,d) \arrow[d, mapsto, "{(f,g)}"] & B(c,d) \arrow[r, "\eta_{(c,d)}"] \arrow[d, mapsto, "{B(f,g)}"] & T(c,d) \arrow[d, mapsto, "{T(f,g)}"] \\ (c',d') & B(c',d') \arrow[r, "\eta_{(c',d')}"'] & T(c',d') \end{tikzcd}$$

commutes. Now

$$\begin{aligned} T(f,g) \circ \eta_{(c,d)} &= T(f, id_{d'}) T(id_c, g) \eta_{(c,d)} \\ &= T(f, id_{d'}) \eta_{(c,d')} B(id_c, g) \\ &= \eta_{(c',d')} B(f, id_{d'}) B(id_c, g) \\ &= \eta_{(c',d')} B(f,g) \end{aligned}$$

$2 \Leftrightarrow 3$: Assume that $\eta_{(c,-)}$ and $\eta_{(-,d)}$ are natural transformations for all objects c, d in categories $\mathbf{C}$ and $\mathbf{D}$, respectively. We need to see that $\tilde{\eta}$ is a natural transformation between the transposed functors. Let $f : c \to c'$ be a morphism in $\mathbf{C}$. We need to show that the diagram

$$\begin{tikzcd} c \arrow[d, mapsto, "f"] & B_c \arrow[r, Rightarrow, "\eta_{(c,-)}"] \arrow[d, Rightarrow, "\tilde{B}(f)"] & T_c \arrow[d, Rightarrow, "\tilde{T}(f)"] \\ c' & B_{c'} \arrow[r, Rightarrow, "\eta_{(c',-)}"'] & T_{c'} \end{tikzcd}$$

commutes. This holds, since the following diagram

$$\begin{tikzcd} c \arrow[d, mapsto, "f"] & B(c,d) \arrow[r, "\eta_{(c,d)}"] \arrow[d, mapsto, "{B(f,id_d)}"] & T(c,d) \arrow[d, "{T(f,id_d)}"] \\ c' & B(c',d) \arrow[r, "\eta_{(c',d)}"'] & T(c',d) \end{tikzcd}$$

commutes for all objects d in $\mathbf{D}$. Furthermore, the commutativity of the two diagrams above is equivalent with condition 2 and hence we have $3 \Rightarrow 2$.

□

Theorem 3.26 motivates the following convention: If $\eta : F \cong G : \mathbf{C} \to \mathbf{D}$ is a natural isomorphism, we say that $F(c) \cong G(c)$ naturally in c, when the functors F and G and the naturality of isomorphism η_c is clear from the context. We may have situations where F and G are bifunctors and then naturality in (a, a') is equivalent with naturality in variables a and a'.

Theorem 3.27. *Assume that $F : \boldsymbol{C} \to \boldsymbol{D}$ is a functor, where $\boldsymbol{C}$ and $\boldsymbol{D}$ are locally small categories. Then F defines a natural map between $\boldsymbol{C}(a, b)$ and $\boldsymbol{D}(Fa, Fb)$. Additionally, if F is fully faithful, then the restriction to isomorphisms is also a natural isomorphism.*

Proof. Since F is a functor, it defines a map $\eta_{(a,b)} := F_{(a,b)} : \mathbf{C}(a, b) \to \mathbf{D}(Fa, Fb)$ for objects a and b in $\mathbf{C}$. For naturality we need to show that the diagram

$$\begin{tikzcd} \forall(c', d) \arrow[d, "\forall(f^{op}, g)"] & \mathbf{C}(c', d) \arrow[r, "\eta_{(c',d)}"] \arrow[d, "\mathbf{C}(f^{op}, g)"] & \mathbf{D}(Fc', Fd) \arrow[d, "\mathbf{D}\circ(F^{op}\times F)(f^{op}, g)"] \\ \forall(c, d') & \mathbf{C}(c, d') \arrow[r, "\eta_{(c,d')}"'] & \mathbf{D}(Fc, Fd') \end{tikzcd}$$

commutes. Fix the morphisms f and g in $\mathbf{C}$ as above. Let $T \in \mathbf{C}(c', d)$. Now

$$\begin{aligned} F \circ \mathbf{C}(f^{op}, g)(T) &= F(gTf) \\ &= F(g)F(T)F(f) \\ &= \mathbf{D}(F(f)^{op}, F(g))(F(T)) \\ &= \mathbf{D} \circ (F^{op} \times F)(f^{op}, g) \circ F(T). \end{aligned}$$

Thus η, which is defined by the functor F, is a natural transformation. Assume that F is fully faithful. The natural transformation η becomes a natural isomorphism. Since F strictly creates isomorphisms, the restriction of F to the maximal groupoid defines a natural isomorphism between the collections of isomorphisms. □

3.2.2 Comma category

We define a category called comma category. Comma categories encompass many different topics in category theory such as arrow categories.

Definition 3.28. Let $F : \mathbf{C} \to \mathbf{E}$ and $G : \mathbf{D} \to \mathbf{E}$ be functors. We define the comma category over F and G, denoted $F \downarrow G$:

1. The objects are tuples (c, d, k) where c and d are objects in $\mathbf{C}$ and $\mathbf{D}$, respectively, and $k : Fc \to Gd$ is a morphism in $\mathbf{E}$.

2. A morphism $\alpha : (c, d, k) \to (c', d', k')$ is a pair of morphisms (α_1, α_2), where $\alpha_1 : c \to c'$ in $\mathbf{C}$ and $\alpha_2 : d \to d'$ in $\mathbf{D}$ and the diagram

$$\begin{tikzcd} Fc \arrow[r, "k"] \arrow[d, "F\alpha_1"] & Gd \arrow[d, "G\alpha_2"] \\ Fc' \arrow[r, "k'"'] & Gd' \end{tikzcd}$$

   commutes.

3. The composition is the usual componentwise composition . It is well-defined, since attaching these commutative squares creates a commutative rectangle.

This definition yields a category, since the identities exist and the composition is associative. The comma category generalizes the arrow categories, since the arrow category of $\mathbf{C}$ is the same as the comma category $I_{\mathbf{C}} \downarrow I_{\mathbf{C}}$. The slice $\mathbf{C}/c$ and coslice $c/\mathbf{C}$ categories are isomorphic to $I_{\mathbf{C}} \downarrow \overline{c}$ and $\overline{c} \downarrow I_{\mathbf{C}}$, respectively, where $\overline{c} : \mathbf{1} \to \mathbf{C}$ is attained by exponential transpose from the object c.

There exist canonical forgetful functors $H_1 : F \downarrow G \to \mathbf{C}$, $H_2 : F \downarrow G \to \mathbf{D}$ and $H_3 : F \downarrow G \to \mathrm{Ar}(\mathbf{E})$ defined by the following diagram:

$$\begin{tikzcd} F\downarrow G \arrow[r, "H_2"] \arrow[d, "H_1"'] \arrow[dr, "H_3"] & \mathbf{D} \\ \mathbf{C} & Ar(\mathbf{E}) \end{tikzcd} \qquad \begin{tikzcd} (c,d,k) \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & d \\ c & k \end{tikzcd} \qquad \begin{tikzcd} (c,d,k) \arrow[r, "{(f_1,f_2)}"] & (c',d',k') \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & f_2 : d \to d' \\ & f_1 : c \to c' & (Ff_1, Gf_2) : k \to k' \end{tikzcd}$$

There exists a canonical natural transformation $\theta : FH_1 \Rightarrow GH_2 : F \downarrow G \to \mathbf{E}$ where $\theta_{(c,d,k)} = k$. The collection θ of morphisms is a natural transformation because if $\alpha : (c,d,k) \to (c',d',k')$ is a morphism in $F \downarrow G$, the diagram

$$\begin{tikzcd} Fc \arrow[r, "k"] \arrow[d, "F\alpha_1"'] & Gd \arrow[d, "F\alpha_2"] \\ Fc' \arrow[r, "k'"'] & Gd' \end{tikzcd}$$

commutes. Changing notation of the diagram above we see that the following diagram commutes:

$$\begin{tikzcd} FH_1(c,d,k) \arrow[r, "\theta_{(c,d,k)}"] \arrow[d, "FH_1\alpha"'] & GH_2(c,d,k) \arrow[d, "FH_2\alpha"] \\ FH_1(c',d',k') \arrow[r, "\theta_{(c',d',k')}"'] & GH_2(c',d',k') \end{tikzcd}$$

We are motivated to give the following definitions:

Definition 3.29 (Category of comma diagrams over a pair of functors)**.** Let $F : \mathbf{C} \to \mathbf{E}$ and $G : \mathbf{D} \to \mathbf{E}$ be functors. We define the category of comma diagrams over the pair of functors (F, G):

- Let $\mathbf{A}$ be a category with functors $L : \mathbf{A} \to \mathbf{C}$ and $R : \mathbf{A} \to \mathbf{D}$. Let $\eta : F \circ L \Rightarrow G \circ R : \mathbf{A} \to \mathbf{E}$ be a natural transformation. We then call the drawing

$$\begin{tikzcd} \mathbf{A} \arrow[r, "R"] \arrow[d, "L"'] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\eta"] & \mathbf{E} \end{tikzcd}$$

  a comma diagram over the pair of functors (F, G).

- Let the following be comma diagrams:

$$\begin{tikzcd} \mathbf{A} \arrow[r, "R"] \arrow[d, "L"'] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\eta"] & \mathbf{E} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{B} \arrow[r, "R'"] \arrow[d, "L'"'] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "F"] \arrow[ur, Rightarrow, "\eta'"] & \mathbf{E} \end{tikzcd}$$

  We call a functor $T : \mathbf{A} \to \mathbf{B}$ a comma diagram morphism between the above comma diagrams, if the diagrams

$$\begin{tikzcd} \mathbf{A} \arrow[dr, "T"] \arrow[drr, bend left, "R"] \arrow[ddr, bend right, "L"'] & & \\ & \mathbf{B} \arrow[r, "R'"] \arrow[d, "L'"] & \mathbf{D} \\ & \mathbf{C} & \end{tikzcd} \qquad \begin{tikzcd} F \circ L \arrow[r, Rightarrow, bend left, "\eta"] \arrow[r, Rightarrow, bend right, "\eta' * T"'] & G \circ R : \mathbf{A} \to \mathbf{E} \end{tikzcd}$$

  commute.

- The composition of comma diagram morphisms is the usual composition of functors, which is well-defined by the associativity of the horizontal composition of natural transformations.

Theorem 3.30. *Let $F : \boldsymbol{C} \to \boldsymbol{E}$ and $G : \boldsymbol{D} \to \boldsymbol{E}$ be functors. Then the comma category $F \downarrow G$ with the canonical forgetful functors and natural transformation*

$$\begin{tikzcd} F \downarrow G \arrow[r, "H_2"] \arrow[d, "H_1"'] & \boldsymbol{D} \arrow[d, "G"] \\ \boldsymbol{C} \arrow[r, "F"'] \arrow[ur, phantom, "\overset{\theta}{\nearrow}"] & \boldsymbol{E} \end{tikzcd}$$

is the terminal object of the category of comma diagrams over functors F and G.

Proof. Fix a diagram of functors and a natural transformation:

$$\begin{tikzcd} \mathbf{A} \arrow[r, "R"] \arrow[d, "L"'] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "F"'] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}"] & \mathbf{E} \end{tikzcd}$$

We need to show that there exists a unique functor $T : \mathbf{A} \to F \downarrow G$ where the diagrams

$$\begin{tikzcd} \mathbf{A} \arrow[dr, dotted, "\exists ! T"] \arrow[drr, bend left, "R"] \arrow[ddr, bend right, "L"'] & & \\ & F \downarrow G \arrow[r, "H_2"] \arrow[d, "H_1"] & \mathbf{D} \\ & \mathbf{C} & \end{tikzcd} \qquad \begin{tikzcd} FL \arrow[r, Rightarrow, bend left, "\alpha"] \arrow[r, Rightarrow, bend right, "\theta T"'] & GR \end{tikzcd}$$

commute. If such a functor $T : \mathbf{A} \to F \downarrow G$ exists, then by the commutative diagrams we have $T(a) = (L(a), R(a), \alpha_a)$. Additionally, fixing $f : a \to b$ in $\mathbf{A}$ it follows that $T(f) = (L(f), R(f)) : (L(a), R(a), \alpha_a) \to (L(b), R(b), \alpha_b)$. Therefore T is uniquely defined. The previous equalities also define T. Since α is a natural transformation $FL \Rightarrow GR$, T is well-defined and makes the diagrams above commute. □

Theorem 3.31. *Let $\boldsymbol{C}, \boldsymbol{D}$ and $\boldsymbol{E}$ be small categories. Then we have a functor*

$$\downarrow : [\boldsymbol{C}, \boldsymbol{E}]^{op} \times [\boldsymbol{D}, \boldsymbol{E}] \to \boldsymbol{Cat}, \textit{ where } (F, G) \mapsto F \downarrow G.$$

Proof. For natural transformation

$$\begin{cases} \eta : F \Rightarrow F' : \mathbf{C} \to \mathbf{E}, \\ \theta : G \Rightarrow G' : \mathbf{D} \to \mathbf{E}, \end{cases}$$

we need to define a functor $F' \downarrow G \to F \downarrow G'$. We have the following universal diagrams of comma categories:

$$\begin{tikzcd} F' \downarrow G \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathbf{D} \arrow[d, "G"] \\ \mathbf{C} \arrow[r, "F'"] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}"] & \mathbf{E} \end{tikzcd} \qquad \begin{tikzcd} F \downarrow G' \arrow[r, "Q_2"] \arrow[d, "Q_1"'] & \mathbf{D} \arrow[d, "G'"] \\ \mathbf{C} \arrow[r, "F"] \arrow[ur, phantom, "\overset{\beta}{\nearrow}"] & \mathbf{E} \end{tikzcd}$$

Now we attain the following natural transformation, call it $\gamma : FP_1 \Rightarrow G'P_2$:

$$FP_1 \xRightarrow{\eta * P_1} F'P_1 \xRightarrow{\alpha} GP_2 \xRightarrow{\theta * P_2} G'P_2$$

Hence we have the comma diagram

$$\begin{tikzcd} F' \downarrow G \arrow[r, "P_2"] \arrow[d, "P_1"'] & \mathbf{D} \arrow[d, "G'"] \\ \mathbf{C} \arrow[r, "F"] \arrow[ur, phantom, "\overset{\gamma}{\nearrow}"] & \mathbf{E} \end{tikzcd}$$

Applying the universal property of $F \downarrow G'$ we attain the corresponding unique functor $F' \downarrow G \to F \downarrow G'$ and call it $\eta^{op} \downarrow \theta$. We have the following commutative diagrams:

$$\begin{tikzcd}
F' \downarrow G \arrow[dr, dashed, "\eta^{op}\downarrow\theta"] \arrow[drr, bend left, "P_2"] \arrow[ddr, bend right, "P_1"'] & & \\
& F \downarrow G' \arrow[r, "Q_2"] \arrow[d, dashed, "Q_1"] & \mathbf{D} \\
& \mathbf{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
FP_1 \arrow[r, Rightarrow, bend left, "\gamma=(\eta*P_2)\bullet\alpha\bullet(\theta*P_1)"] \arrow[r, Rightarrow, bend right, "\beta*(\eta^{op}\downarrow\theta)"'] & G'P_2
\end{tikzcd}$$

From the commutative diagrams above, we see that $\eta^{op} \downarrow \theta$ maps objects and morphisms as follows:

$$\begin{cases} (c,d,k) \mapsto (c,d,\theta_d \circ k \circ \eta_c) \\ (c,d,k) \xrightarrow{x} (c',d',k') \mapsto (c,d,\theta_d \circ k \circ \eta_c) \xrightarrow{x} (c',d',\theta_{d'} \circ k' \circ \eta_{c'}). \end{cases}$$

Lastly we need to check that $\downarrow$ takes identities to identities and respects composition. It is immediate that $I_F^{op} \downarrow I_G$ is the identity functor on $F \downarrow G$. Let $F \overset{\eta}{\Rightarrow} F' \overset{\eta'}{\Rightarrow} F'' : \mathbf{C} \to \mathbf{E}$ and $G \overset{\theta}{\Rightarrow} G' \overset{\theta'}{\Rightarrow} G'' : \mathbf{D} \to \mathbf{E}$ be natural transformations. We need to see that

$$\downarrow ((\eta^{op}, \theta') \circ (\eta'^{op}, \theta)) = \downarrow (\eta^{op}, \theta') \circ \downarrow (\eta'^{op}, \theta)$$

This follows from the terminality of comma category and the fact that both of the functors in the equation above are comma diagram morphisms. □

The following theorem tells us what the dual of a comma category looks like.

Theorem 3.32. *Let $F : \boldsymbol{C} \to \boldsymbol{E}$ and $G : \boldsymbol{D} \to \boldsymbol{E}$ be functors. Then the categories $(F \downarrow G)^{op}$ and $G^{op} \downarrow F^{op}$ are isomorphic via the functor that takes $(\alpha_1, \alpha_2)^{op}$ to $(\alpha_2^{op}, \alpha_1^{op})$ for all morphisms (α_1, α_2) in $F \downarrow G$.*

Proof. Consider the universal comma diagram of $F \downarrow G$:

$$\begin{tikzcd}
F \downarrow G \arrow[r, "H_2"] \arrow[d, dashed, "H_1"'] & \mathbf{D} \arrow[d, dashed, "G"] \\
\mathbf{C} \arrow[r, "F"'] \arrow[ur, Rightarrow, "\theta"] & \mathbf{E}
\end{tikzcd}$$

and consider the comma diagram

$$\begin{tikzcd}
(F \downarrow G)^{op} \arrow[r, "H_1^{op}"] \arrow[d, dashed, "H_2^{op}"'] & \mathbf{C}^{op} \arrow[d, dashed, "F^{op}"] \\
\mathbf{D}^{op} \arrow[r, "G^{op}"'] \arrow[ur, Rightarrow, "\theta^{op}"] & \mathbf{E}^{op}
\end{tikzcd}$$

where $\theta^{op} := (\theta_x^{op})_x$ for objects x in $F \downarrow G$. Notice that it also is a terminal comma diagram over the functor pair (G^{op}, F^{op}).

□

3.2.3 Adjointness and equivalence

In this subsection we develop methods to create functors from universal properties. This naturally leads us to the concept of adjoint functors.

Definition 3.33 (Equivalence of categories)**.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We call the functor F an equivalence, and hence the categories $\mathbf{C}$ and $\mathbf{D}$ equivalent, if there exists a functor $G : \mathbf{D} \to \mathbf{C}$ where

$$G \circ F \cong I_{\mathbf{C}} \text{ and}$$
$$F \circ G \cong I_{\mathbf{D}}.$$

Remark 3.34. If a functor F is an equivalence and the functor G is a corresponding inverse equivalence, then F must be fully faithful and dense: Since GF is faithful and FG is dense, it follows that F is dense and faithful by Theorem 3.24. Similarly the functor G is dense. Because FG is full and G dense, by Theorem 2.28, F is full.

Example 3.35. Let L and T be the vocabulary and the theory of categories. Now the canonical functor $U : \mathbf{Cat} \to \mathbf{Model}_L^T$ is a retraction and an equivalence of categories.

This relation, being equivalent, between categories is an equivalence relation, since reflexivity, and symmetry and transitivity hold. The transitivity is seen by the functoriality of the horizontal composition $*$.

Definition 3.36 (Adjoint functor)**.** Let $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{C}$ be functors. We say that F is the left adjoint to the functor G, if there exists a natural transformation $\lambda : I_{\mathbf{C}} \Rightarrow GF : \mathbf{C} \to \mathbf{C}$ where $\lambda_c : c \to GFc$ is an initial object in the category $c \downarrow G$ for all objects c in $\mathbf{C}$:

$$\begin{tikzcd} c \arrow[r, "\forall f"] \arrow[d, "\lambda_c"'] & Gd, \forall d \\ GFc \arrow[ur, "G(g), \exists ! g : Fc \to d"'] & \end{tikzcd}$$

We denote this by $F \dashv G$ with unit λ.

Example 3.37. Let M be an R-module and assume that M has a basis B. The Basis Theorem of Linear Algebra states that for any R-module N and a function $f : B \to N$ there exists a unique linear extension $\tilde{f} : M \to N$ where the diagram

$$\begin{tikzcd} B \arrow[r, "\forall f"] \arrow[d, hook, "i"'] & \forall N \\ M \arrow[ur, "\exists ! \text{ linear } \tilde{f}"'] & \end{tikzcd}$$

commutes. To dress this in a more categorical formulation, denote the forgetful functor from **R-Mod** to **Set** by G. Now we have exactly the following commutative diagram

$$\begin{tikzcd} B \arrow[r, "\forall f"] \arrow[d, hook, "i"'] & G(N), \forall N \\ G(M) \arrow[ur, "G(\tilde{f}), \exists ! \tilde{f} : M \to N"'] & \end{tikzcd}$$

Therefore $i : B \hookrightarrow G(M)$ is an initial object in $B \downarrow G$. This hints at a new categorical definition of a basis object: Given a functor $G : \mathbf{D} \to \mathbf{C}$, then a triple (c, d, k) is a basis object of d, if (c, d, k) is an initial object in $c \downarrow G$. Dually we have a cobasis object (d, c, k) if and only if (d, c, k) terminal object in $G \downarrow c$. In this categorical sense the commutative monoid of natural numbers with addition is a basis object for the group of integers with addition. Similarly the integral domain of integers is a basis object for the field of rational numbers. In both cases the functor $'G'$ is the forgetful one.

Question arises if every object c in $\mathbf{C}$ becomes a basis object for some object d in $\mathbf{D}$ via $k : c \to Gd$, then can we construct a functor $F : \mathbf{C} \to \mathbf{D}$ that is a left adjoint for G. The answer is yes and the functor F is essentially unique. If the functor G is a forgetful functor, then F is called a free construction.

The following functor creation lemma is a very commonly used tool.

Lemma 3.38 (Functor creation)**.** *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. For every object d in $\boldsymbol{D}$ fix an object c_d in $\boldsymbol{C}$. Assume that there is a morphism $\eta_d : d \to Fc_d$ for all objects d in $\boldsymbol{D}$. Assume that for every morphism $g : a \to b$ in $\boldsymbol{D}$ there exists a unique morphism $f : c_a \to c_b$ in $\boldsymbol{C}$ where the diagram*

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "\eta_a"'] & b \arrow[d, "\eta_b"] \\ Fc_a \arrow[r, "F(f)"'] & Fc_b \end{tikzcd}$$

commutes. Then there exists a unique functor $G : \boldsymbol{D} \to \boldsymbol{C}$ where $\eta : I_{\boldsymbol{D}} \Rightarrow FG : \boldsymbol{D} \to \boldsymbol{D}$ and $G(c) = d_c$ for all objects c in $\boldsymbol{C}$.

Proof. The uniqueness is clear. Define $G : \mathbf{D} \to \mathbf{C}$ by

$$\begin{cases} c \mapsto d_c \text{ and} \\ (g : a \to b) \mapsto (f : c_a \to c_b) \end{cases}$$

where f is the unique morphism making the diagram above commute. By assumption G is well-defined as a pair of functions. Now G maps identities to identities, since fixing an object d in $\mathbf{D}$, then $G(id_d)$ is identity on c_d because the identity id_{c_d} is the only morphism that makes the associated diagram commute. Assume that $d \xrightarrow{g} d' \xrightarrow{g'} d''$ are morphisms in $\mathbf{D}$. Then the diagram

$$\begin{tikzcd}
d \arrow[r, "g"] \arrow[d, "\eta_a"] \arrow[rr, bend left, "g'g"] & d' \arrow[r, "g'"] \arrow[d, "\eta_b"] & d'' \arrow[d] \\
FGd \arrow[r, "G(g)"'] \arrow[rr, bend right, "G(g'g)"'] & FGd' \arrow[r, "G(g')"'] & FGd''
\end{tikzcd}$$

commutes. By the uniqueness, $G(g'g) = G(g')G(g)$. Thus G is a functor. □

It's good to notice that if there is a bifunctor $B : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ and a morphism $f : c \to c'$ in $\mathbf{C}$, then $B(f, -)$ defines a natural transformation $B_c \Rightarrow B_{c'} : \mathbf{D} \to \mathbf{E}$. Additionally, if f is an isomorphism, then $B(f, -)$ is a natural isomorphism. We use this notation in the third part of the following lemma:

Lemma 3.39 (Adjoint Creation)**.** *Let $G : \boldsymbol{D} \to \boldsymbol{C}$ be a functor. Then the following is true:*

1. *Let λ_c be an initial object in $c \downarrow G$ for every object c in $\boldsymbol{C}$. Denote $\lambda = (\lambda_c)_c$. Now $\lambda_c : c \to Gd_c$ for some unique object d_c in D, for objects c in $\boldsymbol{C}$. Then there exists a unique functor $F : \boldsymbol{C} \to \boldsymbol{D}, c \mapsto d_c$, where λ becomes a natural transformation $I_{\boldsymbol{C}} \Rightarrow GF : \boldsymbol{C} \to \boldsymbol{C}$. Furthermore we attain a natural transformation $\epsilon : FG \Rightarrow I_{\boldsymbol{D}}$ where the morphisms ϵ_d are uniquely defined by the following commutative diagram:*

$$\begin{tikzcd}
Gd \arrow[r, "id_{Gd}"] \arrow[d, "\lambda_{Gd}"'] & Gd \\
GFGd \arrow[ur, "G(\epsilon_d)"'] &
\end{tikzcd} \tag{3.4}$$

   and ϵ_d is a terminal object in $F \downarrow d$ for all objects d in $\boldsymbol{D}$. We call ϵ a counit of $F \dashv G$ and say the pair (λ, ϵ) is a unit-counit pair of the adjoint $F \dashv G$.

2. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor and $\lambda : I_{\boldsymbol{C}} \Rightarrow GF$ and let $\epsilon : FG \Rightarrow I_D$ be natural transformations. Then the following conditions are equivalent:*

   (a) *The following pair of unit-counit equations hold:*

$$\begin{cases} \epsilon F \bullet F\lambda = F, \\ G\epsilon \bullet \lambda G = G. \end{cases} \tag{3.5}$$

   (b) *The morphism λ_c is initial in $c \downarrow G$ for all objects c in $\boldsymbol{C}$ and the diagram* (3.4) *commutes.*

   (c) *The morphism ϵ_d is terminal in $F \downarrow d$ for all objects d in $\boldsymbol{D}$ and the dualized diagram of* (3.4) *commutes. Specifically the diagram*

$$\begin{tikzcd}
 & FGFc \arrow[d, "\epsilon_{Fc}"] \\
Fc \arrow[ur, "F\lambda_c"] \arrow[r, "id_{Fc}"'] & Fc
\end{tikzcd} \tag{3.6}$$

   commutes.

3. *Let (λ, ϵ) be a unit-counit pair of an adjoint $F \dashv G$. Let $\boldsymbol{I}$ be a category and denote $G * (-) : [\boldsymbol{I}, \boldsymbol{D}] \to [\boldsymbol{I}, \boldsymbol{C}]$. Then $(\lambda * (-), \epsilon * (-))$ is a unit-counit pair of the adjoint $F * (-) \dashv G * (-)$. Especially (F, λ) is an initial object in $id_{\boldsymbol{C}} \downarrow G * (-)$ when $\boldsymbol{I}$ is chosen to be $\boldsymbol{C}$. Hence the left or right adjoint for a fixed functor is unique up to a canonical isomorphism.*

Proof.

1. Assume that $\lambda_c : c \to Gd_c$ is an initial object in $c \downarrow G$ for all objects c in $\mathbf{C}$. Using the Functor Creation Lemma 3.38 we are going to construct the functor F. It suffices to show that for any $f : c \to c'$ in $\mathbf{C}$ there exists a unique morphism $g : d_c \to d_{c'}$ in $\mathbf{D}$ where the diagram

$$\begin{tikzcd} \forall c \arrow[r, "\forall f"] \arrow[d, dashed, "\lambda_c"] & \forall c' \arrow[d, dashed, "\lambda_{c'}"] & d_c \arrow[d, dashed, "\exists! g"] \\ Gd_c \arrow[r, "Gg"'] & Gd_{c'} & d_{c'} \end{tikzcd}$$

commutes. Hence fix a morphism $f : c \to c'$. Thus we attain the diagram

$$\begin{tikzcd} c \arrow[r, "\lambda_{c'} \circ f"] \arrow[d, dashed, "\lambda_c"] & Gd_{c'} \\ Gd_c & \end{tikzcd}$$

and hence, by the initiality of λ_c, we have a unique morphism $g : d_c \to d_{c'}$ making the diagram commute. By the Functor Creation Lemma there exists a unique functor $F : \mathbf{C} \to \mathbf{D}$ where $\lambda : I_{\mathbf{C}} \Rightarrow GF$ and $Fc = d_c$ for all objects c in $\mathbf{C}$.

Again, by the initiality of λ_{Gd}, there exists a unique morphism $\epsilon_d : FGd \to d$ such that the diagram (3.4) commutes. Fix an object d in $\mathbf{D}$. We will show that ϵ_d is terminal in $F \downarrow d$. Let $g : Fc \to d$ be an object in $F \downarrow d$. We need to show that there exists a unique $f : c \to Gd$ where the diagram

$$\begin{tikzcd} & FGd \arrow[d, "\epsilon_d"] \\ Fc \arrow[ur, "Ff"] \arrow[r, "g"'] & d \end{tikzcd}$$

commutes.

Uniqueness: If such f exists, then

$$\begin{aligned} Gg \circ \lambda_c &= G(\epsilon_d \circ Ff)\lambda_c \\ &= G(\epsilon_d)G(Ff)\lambda_c \\ &= G(\epsilon_d)\lambda_{Gd} f \\ &= id_{Gd} f \\ &= f. \end{aligned}$$

Therefore the uniqueness is clear.

Existence: Define $f = Gg \circ \lambda_c$. To show that $\epsilon_d \circ F(f) = g$, it suffices that the diagram

$$\begin{tikzcd} c \arrow[r, "Gg \circ \lambda_c"] \arrow[d, "\lambda_c"] & Gd \\ GFc \arrow[ur, "G(T)"'] & \end{tikzcd}$$

commutes for both cases $T = g$ and $T = \epsilon_d \circ Ff$. The diagram clearly commutes with $T = g$. For an arbitrary $f' : Gd \to c$, it follows that

$$\begin{aligned} G(\epsilon_d \circ Ff')\lambda_c &= G(\epsilon_d)GFf' \circ \lambda_c \\ &= G(\epsilon_d)\lambda_{Gd} \circ f' \\ &= id_{Gd} \circ f' \\ &= f'. \end{aligned}$$

Thus

$$G(\epsilon_d \circ Ff)\lambda_c = f = G(g)\lambda_c.$$

Hence ϵ_d is terminal in $F \downarrow d$. Fixing any object c in $\mathbf{C}$, it follows that the diagram

$$\begin{tikzcd} & FGFc \arrow[d, "\epsilon_{Fc}"] \\ Fc \arrow[ur, "F(\lambda_c)"] \arrow[r, "id_{Fc}"'] & Fc \end{tikzcd}$$

commutes by the fact that

$$G(\epsilon_d \circ F\lambda_c)\lambda_c = \lambda_c = G(id_F c)\lambda_c.$$

By duality it follows that ϵ is a natural transformation $FG \Rightarrow I_{\mathbf{D}} : \mathbf{D} \to \mathbf{D}$. To be more specific, ϵ_d^{op} is initial in $d \downarrow F^{op}$. Therefore we may use the previous argument to find the unique functor with respect to whom ϵ^{op} becomes a natural transformation. This clearly is G^{op}. Therefore ϵ is a natural transformation $FG \Rightarrow I_{\mathbf{C}}$.

2. It suffices to show that $(a) \Rightarrow (b), (b) \Leftrightarrow (c)$, and (b) and $(c) \Rightarrow (a)$. By the previous part we have seen that $(b) \Rightarrow (c)$ and by duality it follows that $(c) \Rightarrow (b)$. The diagrams (3.4) and (3.6) together say exactly that the equations of (3.5) hold. Thus (b) and $(c) \Rightarrow (a)$.

   To finish the proof we will show $(a) \Rightarrow (b)$: Assume that the equations (3.5) hold. The second equation of (3.5) yields immediately that the diagram (3.4) commutes for all objects d in $\mathbf{D}$. It remains to show that λ_c is initial in $c \downarrow G$ for all objects c in $\mathbf{C}$. Let $f : c \to Gd$ be a morphism in $\mathbf{C}$. We need to show that there exists a unique morphism $g : Fc \to d$ in $\mathbf{D}$ where the diagram

$$\begin{tikzcd} \forall c \arrow[r, "\forall f"] \arrow[d, "\lambda_c"] & Gd, \forall d & & Fc \arrow[d, "\exists! g"] \\ GFc \arrow[ur, "G(g)"'] & & & d \end{tikzcd}$$

   commutes. The Existence of g follows from the second unit-counit equation and the uniqueness from the first: We define

$$g = \epsilon_d \circ Ff : Fc \to FGd \to d.$$

   Using the second equation, we have

$$\begin{aligned} G(g)\lambda_c &= G(\epsilon_d \circ Ff)\lambda_c \\ &= G(\epsilon_d)GFf \circ \lambda_c \\ &= G\epsilon_d \circ \lambda_{Gd} f \\ &= (G\epsilon \bullet \lambda G)_d f \\ &= f. \end{aligned}$$

   The uniqueness is seen as follows: Assume that $g' : Fc \to d$ satisfies $G(g')\lambda_c = f$. Then $g = \epsilon_d \circ Ff = g'$ because

$$\begin{aligned} \epsilon_d \circ Ff &= \epsilon_d \circ F(G(g') \circ \lambda_c) \\ &= \epsilon_d \circ FGg' \circ F\lambda_c \\ &= g' \circ \epsilon_{Fc} \circ F\lambda_c \\ &= g'(\epsilon F \bullet F\lambda)_c \\ &= g' id_{Fc} \\ &= g'. \end{aligned}$$

3. We will denote by F', G' the functors $F * (-), G * (-)$ and with λ', ϵ' we denote the natural transformations $\lambda * (-) : id_{[\mathbf{I},\mathbf{C}]} \Rightarrow G'F' : [\mathbf{I},\mathbf{C}] \to [\mathbf{I},\mathbf{C}]$ and $\epsilon * (-) : F'G' \Rightarrow id_{[\mathbf{I},\mathbf{D}]}$:

$[\mathbf{I}, \mathbf{D}] \to [\mathbf{I}, \mathbf{D}]$. We show that the unit-counit equations hold. Let $D : \mathbf{I} \to \mathbf{D}$ be a functor. Now

$$\begin{aligned}(\epsilon' F \bullet F'\lambda)_D &= \epsilon'_{F'(D)} \bullet F'(\lambda'_D) \\ &= \epsilon F D \bullet F \lambda D \\ &= (\epsilon F \bullet F\lambda) D \\ &= FD \\ &= F'_D.\end{aligned}$$

Thus $\epsilon' F' \bullet F' \lambda = F'$. Similarly $G' \epsilon' \bullet \lambda' G' = G'$ and therefore the pair (λ', ϵ') is a unit-counit pair of the adjoint $F' \dashv G'$.

To see the essential uniqueness of the left adjoint of G, we choose $\mathbf{I} = \mathbf{C}$. Now F is an object of $[\mathbf{C}, \mathbf{D}]$ and thus the pair $(F * (id_{[\mathbf{C},\mathbf{C}]}), \lambda * (id_{[\mathbf{C},\mathbf{C}]})) = (F, \lambda)$ is the initial object of $id_{[\mathbf{C},\mathbf{C}]} \downarrow G * (-)$, which shows that G has an essentially unique left adjoint functor.

□

For any unit natural transformation any corresponding counit is uniquely defined and dually a unit is unique if a counit is fixed.

The Adjoint Creation Lemma states that if $G : \mathbf{D} \to \mathbf{C}$ is a functor and if every object c becomes a basis of some object in $\mathbf{D}$ with respect to the functor G, then G is a right adjoint functor for essentially unique functor $F : \mathbf{C} \to \mathbf{D}$. The Functor Creation Lemma and Adjoint Creation Lemma construct the functors.

Example 3.40.

1. Let P and Q be prosets and $f : P \rightleftarrows Q : g$ increasing maps. The pair (f, g) is called a Galois connection if the pair becomes an adjoint pair as functors. If $f \dashv g$, we have by definition that $c \leq gf(c)$, and moreover if $c \leq g(d)$, then $f(c) \leq d$ for all objects c and d in P and Q, respectively. This is equivalent with the condition that

   $f(c) \leq d$ if and only if $c \leq g(d)$ for all objects c and d in P and Q, respectively.

   (a) Fix a topological space X and denote the set of closed sets of X by $\mathcal{F}$. The closure function $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{F}$ is the left adjoint for the inclusion function $\mathcal{F} \hookrightarrow \mathcal{P}(X)$.

   (b) The right adjoint to the inclusion $\tau \hookrightarrow \mathcal{P}(X)$ of the topology of space X to the power set is the interior function.

   (c) Let $f : X \to Y$ be a function. It defines three set functions $f_*, f_! : \mathcal{P}(X) \rightleftarrows \mathcal{P}(Y) : f^*$, where f_* is the direct image map, f^* is the inverse image map and $f_!(A) = Y \setminus f[X \setminus A]$ for $A \subset X$. Now
   $$f_* \dashv f^* \dashv f_!.$$

2. A functor F is called a free functor or a free construction, if it is a left adjoint for some forgetful functor $G : \mathbf{D} \to \mathbf{C}$.

   (a) Since every set becomes a basis of an R-module, it follows that the forgetful functor from the category **R-Mod** of R-modules is a right adjoint functor. The functor that creates these free R-modules is then a free functor.

   (b) The construction from commutative monoids to groups is a free construction.

   (c) We have a free functor Free : **Multigraph** $\to$ **Cat**, which takes a multigraph G to the free category Free(G) over G.

Now we are going to see how these creation lemmas are used to define new functors.

Theorem 3.41. *The following claims hold for a category* $\mathbf{C}$*:*

1. *Assume that* $\mathbf{C}$ *is a category with binary product objects. Then there exists a functor* $\times : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ *that is a right adjoint to the diagonal functor* $\Delta = (I_{\mathbf{C}}, I_{\mathbf{C}}) : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$*, called the categorical product bifunctor.*

2. *Dually if $\boldsymbol{C}$ has coproducts, we have a left adjoint for the diagonal functor $\Delta : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$, called the coproduct bifunctor.*

3. *If $\boldsymbol{C}$ has products and exponential objects, it follows that for an object a in $\boldsymbol{C}$ we get a right adjoint for the functor $a \times (-)$ called the exponential functor of a, denoted by $(-)^a$ and $[a, -]$.*

Proof.

1. By the Adjoint Creation Lemma it suffices to show that there exists a terminal object in $\Delta \downarrow (c, d)$ for all objects c, d in $\mathbf{C}$. Fix c and d in $\mathbf{C}$. Now there exists the product object $c \times d$ with the projections (pr_1, pr_2). Thus we get a morphism $\Delta(c \times d) \xrightarrow{(pr_1, pr_2)} (c, d)$ in $\mathbf{C} \times \mathbf{C}$. Fix any morphism $(f, g) : \Delta(x) \to (c, d)$. Then the universal property of product objects yield that there exists a unique morphism $h : x \to c \times d$ where the diagram

$$\begin{tikzcd} & \Delta(c \times d) \arrow[d, dashed, "(pr_1, pr_2)"] \\ \Delta(x) \arrow[ur, "\Delta(h)"] \arrow[r, "(f,g)"'] & (c, d) \end{tikzcd}$$

commutes. Here h is the morphism that was denoted by (f, g). Especially we notice that the pairs of projections define the counit. Furthermore the diagonal morphisms $(id_c, id_c) : c \to c \times c$ form the corresponding unit.

2. By duality we have the coproduct functor as the left adjoint for Δ.

3. Let a and b be objects in $\mathbf{C}$. We are looking for a terminal object in the category $(-) \times b \downarrow b$. In other words we are looking for an object d and a morphism $f : d \times a \to b$ such that given any other morphism $g : d' \times a \to b$ in $\mathbf{C}$ there exists a unique morphism $h : d' \to d$ where the diagram

$$\begin{tikzcd} & d \times a \arrow[d, dashed, "f"] & d' \arrow[d, dashed, "\exists! h"] \\ d' \times a, \forall d' \arrow[ur, "h \times id_a"] \arrow[r, "\forall g"'] & b & d \end{tikzcd}$$

commutes. This is exactly the same as the universal property of an exponential object (b^a, f).

$\square$

Theorem 3.42. *Let $G : \boldsymbol{D} \to \boldsymbol{C}$ be a functor. Then G is an equivalence of categories, if and only if G is fully faithful and dense.*

Proof. By the remark 3.34 we have already seen that an equivalence is fully faithful and dense. Assume then that G is fully faithful and dense. For every object c in $\mathbf{C}$ fix an object d_c in $\mathbf{D}$ where $Gd_c \cong c$. Fix then an isomorphism $\lambda_c : c \to Gd_c$ for all object c in $\mathbf{C}$. Let c be an object in $\mathbf{C}$. We will show that λ_c is initial in $c \downarrow G$. Let $f : c \to Gd$ be a morphism in $\mathbf{C}$. We need to see that there exists a unique morphism $g : d_c \to d$ making the diagram

$$\begin{tikzcd} c \arrow[r, "f"] \arrow[d, "\lambda_c"'] & Gd \\ Gd_c \arrow[ur, "Gg"'] & \end{tikzcd}$$

commute. Since λ_c is an isomorphism and G is fully faithful, such a unique g exists. Therefore there exists a unique functor $F : \mathbf{C} \to \mathbf{D}$ where $Fc = d_c$ and $\lambda : I_{\mathbf{C}} \Rightarrow GF : \mathbf{C} \to \mathbf{C}$ by the Adjoint Creation Lemma 3.39. Now λ is a natural isomorphism, since every component is an isomorphism. Notice that the counit $\epsilon : FG \Rightarrow I_{\mathbf{D}}$ is also an isomorphism: Since the diagram

$$\begin{tikzcd} Gd \arrow[r, "id_{Gd}"] \arrow[d, "\lambda_{Gd}"'] & Gd \\ GFGd \arrow[ur, "G(\epsilon_d)"'] & \end{tikzcd}$$

commutes, it follows that $G(\epsilon_d)$ is an isomorphism. Thus by the fully faithfulness of G, ϵ_d is an isomorphism for every object d in $\mathbf{D}$. $\square$

By the proof of Theorem 3.42 we attain the following: If there are functors $F : \mathbf{C} \rightleftarrows \mathbf{D} : G$ and natural isomorphisms

$$\begin{cases} I_{\mathbf{C}} \cong GF \\ FG \cong I_{\mathbf{D}} \end{cases}$$

it suffices to change either one of the natural isomorphisms to obtain a unit-counit pair. The existence of unit-counit pair for functors F and G is then a weaker condition than that of an equivalence.

Chapter 4

Representation of functors

In this chapter we study **Set** valued functors and their behavior. Any locally small category **C** defines a **Set**-valued hom-functor $\mathbf{C} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$. These hom-functors have a lot of structure and many functors look like a hom-functor. Soon we will prove an important lemma, Yoneda lemma, that works as a tool to understand arbitrary **Set** valued functors.

4.1 Yoneda Lemma

Definition 4.1. Let **C** be a locally small category. We say that a functor $F : \mathbf{C} \to \mathbf{Set}$ is a representable functor, if there exists an object c in **C**, where $F \cong \mathbf{C}_c$, where $\mathbf{C}_c$ is the component functor of the hom-functor **C**. The object c is then called the representation of F.

We need to justify the usage of the article 'the'. We will show later by Yoneda embedding that the representation of a functor is unique up to a unique isomorphism.

Example 4.2.

1. The identity functor $I_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is represented by a terminal object in **Set**.
2. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is represented by the integers $\mathbb{Z}$: Let G be a group. Now we have a bijection $\mathbf{Grp}(\mathbb{Z}, G) \to U(G), f \mapsto f(1)$. Since this bijection is natural, it is a natural isomorphism and so the forgetful functor U is represented by $\mathbb{Z}$.
3. The contravariant power set functor $\mathcal{P}^* : \mathbf{Set}^{op} \to \mathbf{Set}$ is represented by a two element set.

Definition 4.3. Let **C** be a locally small category. The functor $y : \mathbf{C}^{op} \to [\mathbf{C}, \mathbf{Set}]$, attained from the hom-functor $\mathbf{C} : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{Set}$ via exponential application, is called Yoneda embedding.

Theorem 4.4 (Yoneda lemma)**.** *Let* $\boldsymbol{C}$ *be a locally small category. Then*

$$[\boldsymbol{C}_c, F] \cong Fc, \eta \xmapsto{\Psi_{(c,F)}} \eta_c(id_c)$$

naturally in objects c *and* $F : \boldsymbol{C} \to [\boldsymbol{C}, \boldsymbol{Set}]$*, respectively.*

Proof. Fix an object c in **C** and a functor $F : \mathbf{C} \to \mathbf{Set}$. We need to show that the map $\eta \mapsto \eta_c(id_c)$ is bijective.

Let $\eta : \mathbf{C}_c \Rightarrow F : \mathbf{C} \to \mathbf{Set}$ be a natural transformation. Let $f : c \to d$ be a morphism in **C**. Notice that $\eta_d(f) = F(f)(\eta_c(id_c))$, because

$$\begin{aligned}\eta_d(f) &= \eta_d(C_c(f)(id_c)) \\ &= \eta_d \circ \mathbf{C}_c(f)(id_c) \\ &= Ff \circ \eta_c(id_c) \\ &= Ff(\eta_c(id_c)).\end{aligned}$$

This immediately implies the injectivity of $\Psi_{c,F}$. For the surjectivity, fix an element $x \in F(c)$. Define $\eta_d : \mathbf{C}(c, d) \to F(d)$ by $\eta_d(f) = Ff(x)$. We need to show that the collection of morphisms

η is a natural transformation $\mathbf{C}_c \Rightarrow F$. Let $g : d \to d'$ be a morphism in $\mathbf{C}$. The diagram

$$\begin{tikzcd}
\mathbf{C}(c,d) \arrow[r, "\eta_d"] \arrow[d, "g_*"] & Fd \arrow[d, "Fg"] \\
\mathbf{C}(c,d') \arrow[r, "\eta_{d'}"'] & Fd'
\end{tikzcd}$$

commutes: Now

$$\begin{aligned}
Fg \circ \eta_d(f) &= FgFf(x) \\
&= F(gf)(x) \\
&= \eta_{d'}(gf) \\
&= \eta_{d'} \circ g_*(f).
\end{aligned}$$

Thus η is a natural transformation. Since

$$\Psi_{(c,F)}(\eta) = \eta_c(id_c) = F(id_c)(x) = x,$$

it follows that $\Psi_{(c,F)}$ is bijective.

It remains to show that Ψ is a natural transformation. Let $f : c \to d$ be a morphism in $\mathbf{C}$ and let $\theta : F \Rightarrow G : \mathbf{C} \to \mathbf{Set}$ be a natural transformation. We show that the diagram

$$\begin{tikzcd}
(c,F) \arrow[d, "{(f,\theta)}"] & {[C_c, F]} \arrow[r, "\Psi_{(c,F)}"] \arrow[d, "{[y(f)^{op},\theta]}"] & Fc \arrow[d, "G(f)\theta_c"] \\
(d,G) & {[C_d, G]} \arrow[r, "\Psi_{(d,G)}"'] & Gd
\end{tikzcd}$$

commutes. Let $\eta : \mathbf{C}_c \Rightarrow F$. Now

$$\begin{aligned}
G(f)\theta_c\Psi_{(c,F)}(\eta) &= G(f)\theta_c\eta_c(id_c) \\
&= G(f)(\theta \bullet \eta)_c(id_c) \\
&= (\theta \bullet \eta)_d(f) \\
&= \theta_d\eta_d(f)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{(d,G)} \circ [y(f)^{op}, \theta](\eta) &= \Psi_{(d,G)}(\theta \bullet \eta \bullet y(f)) \\
&= (\theta \bullet \eta \bullet y(f))_d(id_d) \\
&= \theta_d\eta_d \circ f_*(id_d) \\
&= \theta_d\eta_d(f).
\end{aligned}$$

□

Notice that Yoneda lemma states the existence of an isomorphism

$$\mathbf{Set}^{\mathbf{C}} \circ (y^{op} \times id_{[\mathbf{C},\mathbf{Set}]}) \cong Ev : \mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \to \mathbf{Set}.$$

Corollary 4.5 (Yoneda embedding)**.** *Let $\boldsymbol{C}$ be a locally small category. Let $y : \boldsymbol{C}^{op} \to [\boldsymbol{C}, \boldsymbol{Set}]$ be Yoneda embedding. Then y is injective on objects and fully faithful. Especially y is an embedding with respect to the forgetful functor to $\boldsymbol{SET}$ that takes a category to its $U-$moderate set of morphisms.*[1]

Proof. It is clear that y is injective on objects. It suffices to show that y is fully faithful. Fix objects c and d in $\mathbf{C}$. We need to show that y defines a bijection $\mathbf{C}(a,b) \cong \mathbf{Set}^{\mathbf{C}}(C_b, C_a)$. By Yoneda lemma there is a bijection $\psi : [\mathbf{C}_b, \mathbf{C}_a] \cong \mathbf{C}(a,b)$, where $\eta \mapsto \eta_b(id_b)$. It suffices to show that $\psi \circ y \circ \psi = \psi$. Let $\eta : \mathbf{C}_b \Rightarrow \mathbf{C}_a$ and now

$$\begin{aligned}
\psi(y(\psi(\eta))) &= \psi(y(\eta_b(id_b))) \\
&= (y(\eta_b(id_b)))_b(id_b) \\
&= id_b \circ \eta_b(id_b) \\
&= \psi(\eta).
\end{aligned}$$

□

[1] Fixing a universe U, the category **SET** is the category of U^+-small sets.

Corollary 4.6. *Let $\boldsymbol{C}$ be a locally small category, we will denote Yoneda embedding by $y : \boldsymbol{C} \to \boldsymbol{Set}^{\boldsymbol{C}^{op}}$. Then the following claims hold:*

1. *Let a and b be objects in $\boldsymbol{C}$. Then $a \cong b$ if and only if $\boldsymbol{C}^a \cong \boldsymbol{C}^b$. Additionally, the association between isomorphisms is a natural isomorphism.*
2. *Let $F, G : \boldsymbol{D} \to \boldsymbol{C}$ be functors. Then $F \cong G$ if and only if $y \circ F \cong y \circ G$ and the association between the isomorphisms is a natural isomorphism.*

Proof.

1. Since y is a fully faithful functor, it follows from Theorem 3.27 that y defines a natural isomorphism between the sets of isomorphisms.
2. By the part 1 and using the fact that $y*(-)$ is a fully faithful functor, if follows from Theorem 3.21 that $y*(-)$ defines a natural isomorphism between $\text{Isom}(F, G)$ and $\text{Isom}(y \circ F, y \circ G)$ that is natural in F and G.

□

The fact that Yoneda embedding is an embedding is quite profound realization. By Corollary 4.6, there is a characterization for two objects being isomorphic. If c is an object in a locally small category $\mathbf{C}$ and we look closely at the functor $y(c)$, we notice that $y(c)$ consists exactly of the information of morphisms leaving c. Yoneda lemma, hence, implies that to understand the object c completely, it suffices to understand its connections through morphisms to the rest of the category. These connections define the object itself up to a unique isomorphism. To understand an object c it suffices to understand the hom-sets $Hom(c, d)$ for all objects d. To be more syntactic, $Hom(a, b) \cong Hom(a', b)$ naturally in b implies that $a \cong a'$. If $Hom(Fa, b) \cong Hom(F'a, b)$ naturally in a and b, then $F \cong F'$ for functors $F, F' : \mathbf{D} \to \mathbf{C}$ and we have a direct way to construct one isomorphism from the other.

Example 4.7. By Yoneda embedding we are able prove results relating for example to a Cartesian closed category $\mathbf{C}$ with coproducts. Fix objects a, b, c in $\mathbf{C}$. The following isomorphisms are natural in all variables:

$$
\begin{aligned}
(a \times b)^c &\cong a^c \times b^c, \\
(a + b) \times c &\cong (a \times c) + (b \times c), \\
a^{(b+c)} &\cong a^b \times a^c, \\
(a^b)^c &\cong a^{(b \times c)}.
\end{aligned}
$$

We only show the second isomorphism; the rest of the cases are proven by a similar argument. Let x be any object in $\mathbf{C}$. We show that $Hom((a+b) \times c, x) \cong Hom((a \times c) + (b \times c), x)$ naturally in each variable a, b, c, x and this suffices by the fact that Yoneda embedding is fully faithful:

$$
\begin{aligned}
Hom((a + b) \times c, x) &\cong Hom(a + b, x^c) \\
&\cong Hom(a, x^c) \times Hom(b, x^c) \\
&\cong Hom(a \times c, x) \times Hom(b \times c, x) \\
&\cong Hom((a \times c) + (b \times c), x).
\end{aligned}
$$

Since every isomorphism is natural, we get the natural isomorphism

$$(a + b) \times c \cong (a \times c) + (b \times c).$$

Let X be a set. Especially the poset $\mathcal{P}(X)$ has products, sums and exponentials and so does its dual. Thus we get that the equations hold:

$$
\begin{aligned}
(A \cup B) \cap C &= (A \cap C) \cup (B \cap C) \\
(A \cap B) \cup C &= (A \cup C) \cap (B \cup C).
\end{aligned}
$$

Notice that the set of natural numbers $\mathbb{N}$ with poset structure defined by the divides $|$ relation yields a cocartesian closed category. This then implies that

$$(a \times b) + c = (a + c) \times (a + c)$$

where $+$ and $\times$ are the least common multiple and greatest common divisor operations, respectively.

4.2 Category of elements

Let $\mathbf{C}$ be a locally small category and let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. If F is represented by an object c in $\mathbf{C}$, we find, by Yoneda lemma, that it corresponds naturally to some element x in Fc. We show that there is an other way to deduce the representability of the functor F by looking at the pairs (c, x) where c is an object in $\mathbf{C}$ and $x \in Fc$.

Definition 4.8. Let $\mathbf{C}$ be a locally small category. If $F : \mathbf{C} \to \mathbf{Set}$ is a functor. then the category of elements is

$$\int F = \overline{1} \downarrow F,$$

where 1 denotes a terminal object in $\mathbf{Set}$ and $\overline{1} : \mathbf{1} \to \mathbf{Set}$ is the corresponding exponential transposition of the object 1. If F is instead a contravariant functor $\mathbf{C}^{op} \to \mathbf{Set}$, we define

$$\int F = (\overline{1} \downarrow F)^{op} \cong F^{op} \downarrow \overline{1},$$

where $F^{op} : \mathbf{C} \to \mathbf{Set}^{op}$.

In the category of elements the objects can be thought as pairs (c, x) where $x : 1 \to Fc$ which corresponds exactly to a choice of an element. A morphism $f : (c, x) \to (c', x')$ is a morphism $f : c \to c'$ in $\mathbf{C}$ whose associated map Ff maps x to x'.

Assume that a functor $F : \mathbf{C} \to \mathbf{Set}$ is represented by c in $\mathbf{C}$. Then

$$\int F \cong \int \mathbf{C}_c \cong c/\mathbf{C}$$

and thus $\int F$ has an initial object, since the coslice category $c/\mathbf{C}$ has an initial object id_c. Notice that this uses the functoriality of the comma category (Theorem 3.31). This result extends to a canonical bijection with the initial representations and natural isomorphisms.

Theorem 4.9. *Let $\mathbf{C}$ be a locally small category and let $F : \mathbf{C} \to \mathbf{Set}$ be a functor. Let c be an object in $\mathbf{C}$ and $x \in Fc$. Denote the natural transformation that corresponds, via Yoneda lemma, to the pair (c, x) by $\eta : \mathbf{C}_c \Rightarrow F$. Then η is a natural isomorphism if and only if (c, x) is initial in $\int F$.*

Proof. Notice that the initiality of (c, x) means exactly that given any object (d, y) in $\int F$, there exists a unique $f : (c, x) \to (d, y)$. In other words, there exists a unique $f : c \to d$ where $Ff(x) = y$. This means exactly the same as the existence of a unique $f \in \mathbf{C}(c, d)$ such that $\eta_d(f) = Ff(x) = y$. This is equivalent with η_d being a bijection. This equivalence chain holds for any choice of (d, y). Hence η is an isomorphism, if and only if (c, x) is initial in $\int F$. $\square$

By duality, if F were a contravariant functor, then $\int F$ had a terminal object if and only if the corresponding natural transformation were a natural isomorphism.

By the previous characterization we are motivated to define what we mean by a universal property of an object c in a locally small category $\mathbf{C}$ and what we mean by a universal element.

Definition 4.10 (Universal property and universal element)**.** Let $\mathbf{C}$ be a locally small category with an object c in $\mathbf{C}$. Let F be a contravariant or a covariant functor from $\mathbf{C}$ to $\mathbf{Set}$ and let $x \in Fc$. The pair (F, x) is called the universal property of the object c, if the associated natural transformation $\eta_x : \mathbf{C}_c \Rightarrow F$, via Yoneda lemma, is a natural isomorphism. Furthermore, the element x is then called the universal element of the representation c of the functor F.

Suppose that $\mathbf{C}$ is a locally small category with an object c and let $G : \mathbf{D} \to \mathbf{C}$ be a functor. Then the categories $c \downarrow G, \int \mathbf{C}_c \circ G$ and $1 \downarrow \mathbf{C}_c \circ G$ are canonically isomorphic. In the following examples we find out a new way to define the adjointness of a functor pair.

Example 4.11.

1. Assume that R is a ring. The tensor product of R-modules, is an example of universal element: Let V and W be $R-$modules. Define the functor $\mathrm{Bil} = \mathrm{Bil}(V, W; -) : \mathbf{R\text{-}Mod} \to \mathbf{Set}$ by setting $\mathrm{Bil}(V, W; P)$ to be the set of bilinear maps from $V \times W$ to $R-$module P. Additionally, Bil takes an $R-$linear map to the post composition map.

Let (T,x) be an object in $\int \mathrm{Bil}$. The initiality of (T,x) in $\int \mathrm{Bil}$ is equivalent with the following: Given any bilinear map $B : V \times W \to P$, where P is an $R-$module, there exists a unique morphism $L : (T,x) \to (P,B)$. To put it more concretely, there exists a unique linear map $L : T \to P$ making the diagram

$$\begin{tikzcd} V \times W \arrow[r, "B"] \arrow[d, dashed, "x"'] & P \\ T \arrow[ur, "L"'] & \end{tikzcd}$$

commute. This means exactly the same as the pair (T,x) is the tensor product of modules V and W. So the problem of finding the representation for the functor $\mathrm{Bil}(V,W;-)$ becomes the same as finding the tensor product of the spaces V and W. The R-module T is usually denoted as $V \otimes W$ and the universal bilinear map $x : V \times W \to V \otimes W$ is denoted by $\otimes$. Since this tensor product always exists, it holds that $\mathrm{Bil}(V,W;-) \cong \mathbf{R\text{-}Mod}(V \otimes W, -)$.

2. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ is represented by the integers $\mathbb{Z}$, where a group homomorphism $f : \mathbb{Z} \to G$ is associated to the element $f(1) \in G$ by $\eta_G : \mathbf{Grp}(\mathbb{Z}, G) \to U(G)$ for all homomorphisms f and groups G. The universal element is then $\eta_{\mathbb{Z}}(id_{\mathbb{Z}}) = id_{\mathbb{Z}}(1) = 1$.

3. Let (λ, ϵ) be a unit-counit pair of the adjoint $F : C \dashv D : G$. Notice that (Fc, λ_c) is initial in $c \downarrow G$ and thus initial in $\int \mathbf{C}_c \circ G$. So we see that λ_c is the universal element of the object Fc with respect to the functor $\mathbf{C}_c \circ G$. Hence we find the natural isomorphism $\eta_c : \mathbf{D}_{Fc} \cong \mathbf{C}_c \circ G$, where $\eta_{(c,Fc)}(id_{Fc}) = \lambda_c$. Therefore the isomorphism

$$\mathbf{D}(Fc, d) \cong \mathbf{C}(c, Gd)$$

is natural in d for every object c in $\mathbf{C}$. We will show the naturality in c also. By Yoneda lemma,

$$\eta_{(c,d)}(g) = \mathbf{C}_c \circ G(g)(\lambda_c) = G(g) \circ \lambda_c$$

for a morphism $g : Fc \to d$ and an object d in $\mathbf{D}$. Now $(Gd, \epsilon_d : FGd \to d)$ is a terminal object of $F \downarrow d$. By Yoneda lemma there exists a unique corresponding natural isomorphism $\theta_{(-,d)} : \mathbf{C}^{Gd} \Rightarrow D^d \circ F^{op}$, where $\theta_{(Gd,d)}(id_{Gd}) = \epsilon_d$. Hence

$$\mathbf{C}(c, Gd) \cong \mathbf{D}(Fc, d)$$

naturally in c. By Yoneda lemma,

$$\theta_{(c,d)}(f^{op}) = \mathbf{D}^d \circ F^{op}(f^{op})(\theta_{(Gd,d)}(id_{Gd})) = \mathbf{D}^d(Ff^{op})(\epsilon_d) = \epsilon_d \circ Ff.$$

By the compatibility of unit-counit pair (λ, ϵ), it holds that

$$\begin{aligned} \eta_{(c,d)}(\theta_{(c,d)}(f^{op})) &= \eta_{(c,d)}(\epsilon_d \circ Ff) \\ &= G(\epsilon_d \circ Ff) \circ \lambda_c \\ &= G(\epsilon_d) \circ GFf \circ \lambda_c \\ &= G(\epsilon_d)\lambda_{Gd} f \\ &= f \\ &= f^{op}. \end{aligned}$$

Therefore $\eta_{(c,d)}$ is the inverse map of $\theta_{(c,d)}$ and thus $\eta_{(c,d)}$ is natural in both variables c and d.

4.3 Adjunction of adjoints

From the Example 4.11(3) we obtain a new way to view adjoint functors.

Corollary 4.12. *Let $\boldsymbol{C}$ be a locally small category and let $G : \boldsymbol{D} \to \boldsymbol{C}$ be a functor. Assume that for every object c in $\boldsymbol{C}$ there exists an object e_c in $\boldsymbol{D}$ and natural isomorphism*

$$\eta_{(c,d)} : \boldsymbol{D}(e_c, d) \cong \boldsymbol{C}(c, Gd),$$

natural in d. Then there exists a unique functor $F : \boldsymbol{C} \to \boldsymbol{D}$ such that $Fc = e_c$ and η becomes a natural isomorphism

$$\boldsymbol{D}(Fc, d) \cong \boldsymbol{C}(c, Gd) \tag{4.1}$$

in both variables c and d. Additionally, F is a left adjoint to G, where the component λ_c of a unit λ is defined as the universal element of $\eta_{(c,-)}$ for all objects c in $\boldsymbol{C}$.

Proof. Since $\eta_{(c,-)}$ is a natural isomorphism, we attain that the corresponding object (e_c, λ_c), where $\lambda_c = \eta_{(c,e_c)}(id_{e_c})$, is initial in $\int \mathbf{C}_c \circ G = c \downarrow G$. Thus by the Adjoint Creation Lemma 3.39 we have a unique left adjoint F for G, where $Fc = e_c$ and $\lambda : I \Rightarrow GF$. By Example 4.11)(3), η gives the isomorphism in 4.1 and it's natural in both variables c and d. To show the uniqueness, assume that we have an other functor F' where $F'c = e_c$ and

$$\eta_{c,d} : \mathbf{D}(F'c, d) \cong \mathbf{E}(c, Gd)$$

is natural in c and d. Thus there is the following chain of natural isomorphisms in both c and d:

$$\begin{aligned} \mathbf{D}(Fc, d) &\cong \mathbf{C}(c, Gd) \\ &\cong \mathbf{D}(F'c, d). \end{aligned}$$

Notice that the composition is the identity. Since Yoneda embedding $y : \mathbf{D}^{op} \to \mathbf{Set}^{\mathbf{D}}$ is monic and $y \circ F^{op} = y \circ F'^{op}$, it follows that $F = F'$. $\square$

Definition 4.13. Let $F : \mathbf{C} \rightleftarrows: G$ be functors. A natural isomorphism

$$\eta_{(c,d)} : \mathbf{D}(Fc, d) \cong \mathbf{C}(c, Gd)$$

natural in objects c and d, is called an adjunct of the adjoint $F \dashv G$.

There is a bijective canonical correspondence with the adjuncts η, the units λ and the counits ϵ of an adjoint functor pair. The correspondence associates an adjunct η to a unit λ by setting the component λ_c to be the universal element corresponding to the natural isomorphism $\eta_{(c,-)}$. Similarly the component ϵ_d of a counit is given by η as the universal element of the natural isomorphism $\eta^{-1}_{(-,d)}$.

If η is an adjunct of the adjoint pair $F : \mathbf{C} \dashv \mathbf{D} : G$. Then $\eta^{-1,op} := (\eta^{-1,op}_{(c,d)})_{(c,d)}$ defines an adjunct of $G^{op} : \mathbf{D}^{op} \dashv \mathbf{C}^{op} : F^{op}$.

Notice that the adjunct η satisfies the following: Let $g : Fc \to d$ be a morphism in $\mathbf{D}$. Then $\eta_{(c,d)}(g) : c \to Gd$ is the unique morphism making the diagram

$$\begin{tikzcd} & FGd \arrow[d, "\epsilon_d"] \\ Fc \arrow[ur, "F(\eta(g))"] \arrow[r, "g"'] & d \end{tikzcd}$$

commute. This is seen by the fact that $\eta_{(c,d)}(g) = G(g)\lambda_c$.

Theorem 4.14. *The following statements are true:*

1. *Let $F_1 : \boldsymbol{C} \dashv \boldsymbol{D} : F_2$ and $G_1 : \boldsymbol{D} \dashv \boldsymbol{E} : G_2$. Then $F_2F_1 : \boldsymbol{C} \dashv \boldsymbol{E} : G_1G_2$.*

2. *Let $F : \boldsymbol{C} \rightleftarrows: \boldsymbol{D} : G$ be functors. Assume that there exists a collection of bijections $\eta = (\eta_{(c,d)})_{(c,d)}$*

$$\eta_{(c,d)} : \boldsymbol{D}(Fc, d) \cong \boldsymbol{C}(c, Gd)$$

for all objects c and d. Then η is a natural isomorphism if and only if the following condition holds: Let $s : c \to c'$ and $t : d \to d'$ be morphisms in $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively. Assume that $f : Fc \to d$ and $g : Fc' \to d'$ are morphisms in $\boldsymbol{D}$. If either of the diagrams

$$\begin{tikzcd} Fc \arrow[r, "f"] \arrow[d, "Fs"] & d \arrow[d, "t"] & & c \arrow[r, "\eta(f)"] \arrow[d, "s"] & Gd \arrow[d, "Gt"] \\ Fc' \arrow[r, "g"'] & d' & & c' \arrow[r, "\eta(g)"'] & Gd' \end{tikzcd}$$

commutes, then both of them commute. Moreover, the naturality of η follows from the property that the commutation of the left square above implies the commutation of the right square for all such squares.

Proof.

1. There exists the following chain of isomorphisms
$$\mathbf{E}(F_2F_1c, d) \cong \mathbf{D}(F_1c, G_2d) \cong \mathbf{D}(c, G_1G_2d)$$
natural in c and d. This proves that $F_2F_1 \dashv G_1G_2$.

2. Assume that η is natural. Notice that the naturality of η means that for any morphisms $f : c \to c'$ and $g : d \to d'$ in $\mathbf{C}$ and $\mathbf{D}$, respectively, the diagram
$$\begin{tikzcd} {[Fc', d]} \arrow[r, "\eta"] \arrow[d, "{[F(f)^{op}, g]}"'] & {[c', Gd]} \arrow[d, "{[f^{op}, G(g)]}"] \\ {[Fc, d']} \arrow[r, "\eta"'] & {[c, Gd']} \end{tikzcd}$$
commutes. Put differently, $\eta(gTF(f)) = G(g)\eta(T)f$ for all $T : Fc' \to d$.
Fix morphisms $s : c \to c', h : d \to d', f : Fc \to d$ and $g : Fc' \to d'$ for objects c, c' in $\mathbf{C}$ and d, d' in $\mathbf{D}$. Assume first that $tf = gFs$. Now
$$G(t)\eta(f) = \eta(tf) = \eta(gFs) = \eta(g)s.$$

Assume then that the first and the last formulas are equal in the equations above. Then so does the middle equation hold by the naturality of η. By injectivity of the components of η, it holds that $tf = Gfs$.

Assume then that the commutativity of left square implies the commutativity of right square. To check the naturality of η, fix morphisms $f : c \to c', g : d \to d'$ and $T : Fc' \to d$. We need to show that
$$\eta(gTF(f)) = G(g)\eta(T)f.$$
Consider the diagrams
$$\begin{tikzcd} Fc \arrow[r, "TF(f)"] \arrow[d, "id_{Fc}"'] & d \arrow[d, "g"] \\ Fc \arrow[r, "gTF(f)"'] & d' \end{tikzcd} \qquad \begin{tikzcd} Fc \arrow[r, "TFf"] \arrow[d, "Ff"'] & d \arrow[d, "id"] \\ Fc' \arrow[r, "T"'] & d \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\eta(TF(f))"] \arrow[d, "id"'] & Gd \arrow[d, "G(g)"] \\ c \arrow[r, "\eta(gTF(f))"'] & Gd' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "\eta(TF(f))"] \arrow[d, "f"'] & Gd \arrow[d, "id"] \\ c' \arrow[r, "\eta(T)"'] & Gd \end{tikzcd}$$
Since the first two diagrams commute, so do the second two, by assumption. Therefore η is a natural transformation. □

Theorem 4.15. *Let $F : \boldsymbol{C} \times \boldsymbol{D} \to \boldsymbol{E}$ be a bifunctor. Let $(G_c : \boldsymbol{E} \to \boldsymbol{D})_{c \in Obj(\boldsymbol{C})}$ be a collection of functors and let*
$$\eta_{c,d,e} : \boldsymbol{E}(F_c(d), e) \cong \boldsymbol{D}(d, G_c(e))$$
be bijections natural in d and e for every object c in $\boldsymbol{C}$. Then the collection of functors $(G_c)_c$ extends uniquely to a bifunctor $G : \boldsymbol{C}^{op} \times \boldsymbol{E} \to \boldsymbol{D}$ such that the collection $\eta = (\eta_{c,d,e})_{(c,d,e)}$ becomes natural in c also.

Proof. Again, with Yoneda embedding, we have the uniqueness of the extension to the functors $(G_c)_{c \in \mathrm{Obj}(C)}$.

It remains to see the existence. Denote the counit associated to the adjunct $\eta_{(c,-,-)}$ by $\epsilon^c : F_cG_c \Rightarrow I_{\mathbf{E}} : \mathbf{E} \to \mathbf{E}$ for objects c in $\mathbf{C}$. First we are going to define a collection of functors $G^e : \mathbf{C}^{op} \to \mathbf{D}$ where $G^e(c) = G_c(e)$ and for arrows $f : c \to c'$ and $g : d \to d'$, $G_c(g)G^e(f^{op}) = G^{e'}(f^{op})G'_c(g)$ where e runs through all objects in $\mathbf{E}$. By the Bifunctor Decomposition Theorem we attain a functor $G : \mathbf{C}^{op} \times \mathbf{E} \to \mathbf{D}$.

Fix an object e in $\mathbf{E}$, we define $G^e(c) = G_c(e)$. For a morphism $f : c \to c'$, we define $G^{c'}(e)(f) : G_{c'}(e) \to G_c(e)$ to be the unique morphism $h : G_{c'}(e) \to G_c(e)$, where the diagram
$$\begin{tikzcd} F_c(G_{c'}(e)) \arrow[r, "F(h)"] \arrow[d, "{F(f, id_{G_{c'}(e)})}"'] & F_c(G_c(e)) \arrow[d, "\epsilon^c_e"] \\ F_{c'}(G_{c'}(e)) \arrow[r, "\epsilon^{c'}_e"'] & e \end{tikzcd}$$

commutes. Since the morphism ϵ_e^c is a terminal object in $F_c \downarrow e$, it follows that G^e is a well-defined function on morphisms. From the definition we directly see that the identities are mapped to the corresponding identities by G^e. We need to see that G^e respects composition. Let $c \xrightarrow{f_1} c' \xrightarrow{f_2} c''$ be morphisms in $\mathbf{C}$. Denote $h_1 = G^e(f_1), h_2 = G^e(f_2)$ and $h_3 = G^e(f_2 \circ f_1)$. We will show that $h_3 = h_1 \circ h_2$. Consider the diagram

$$\begin{tikzcd}
F_cG_{c''}(e) \arrow[r, "F_c(h_2)"'] \arrow[d, "F(f_1,id)"'] \arrow[rr, bend left, "{F_c(h_3),F_c(h_1h_2)}"] & F_cG_{c'}(e) \arrow[r, "F_c(h_1)"'] \arrow[d, dashed, "F(f_1,id)"] & F_c(G_c)(e) \arrow[d, "\epsilon_e^{c'}"] \\
F_{c'}G_{c''}(e) \arrow[r, "F_{c'}(h_2)"] \arrow[d, "F(f_2,id)"'] & F_{c'}G_{c'}(e) \arrow[r, "\epsilon_e^{c'}"] & c \\
F_{c''}G_{c''}(e) \arrow[urr, "\epsilon_e^{c''}"'] & &
\end{tikzcd}$$

The bottom triangle commutes, by the definition of h_2. The left and the right squares commute by the bifunctoriality of F and the definition of h_1. From the definition of h_3 we see that the whole exterior of the diagram commutes. Since h_3 is unique such a morphism and h_1h_2 also makes the whole exterior commute, it follows that $h_3 = h_1h_2$. Thus G^e is a functor for every object e in $\mathbf{E}$.

Next we will show that the collections of functors $(G^e)_e$ and $(G_c)_c$ are compatible to apply the Bifunctor Decomposition Theorem. Let $f : c \to c'$ and $h : e \to e'$. We need to show that the diagram

$$\begin{tikzcd}
G_{c'}(e) \arrow[r, "g_1"] \arrow[d, dashed, "G_{c'}(h)"] & G_c(e) \arrow[d, dashed, "G_c(h)"] \\
G_{c'}(e') \arrow[r, "g_2"'] & G_c(e')
\end{tikzcd}$$

commutes where $g_1 = G^e(f^{op})$ and $g_2 = G^{e'}(f^{op})$. To show this, it suffices to see that the diagram

$$\begin{tikzcd}
F_cG_{c'}(e) \arrow[rr, "F_c(T)"] \arrow[d, dashed, "F(f,id)"] & & F_cG_c(e') \arrow[d, dashed, "\epsilon_{e'}^c"] \\
F_{c'}G_{c'}(e) \arrow[r, "\epsilon_e^{c'}"'] & e \arrow[r, "h"'] & e'
\end{tikzcd} \tag{4.2}$$

commutes for both $T = G_c(h)g_1$ and $T = g_2G_{c'}(h)$, since ϵ^c is the counit in $F_c \dashv G_c$. For $T = G_c(h)g_1$ we have the following diagram:

$$\begin{tikzcd}
F_cG_{c'}(e) \arrow[r, "F_c(g_1)"] \arrow[d, dashed, "F(f,id)"] & F_cG_c(e) \arrow[r, "F_c(G_c(h))"] \arrow[d, "\epsilon_e^c"] & F_cG_c(e') \arrow[d, dashed, "\epsilon_{e'}^c"] \\
F_{c'}G_{c'}(e) \arrow[r, "\epsilon_e^{c'}"'] & e \arrow[r, "h"'] & e'
\end{tikzcd} \tag{4.3}$$

The left square commutes by the definition of g_1 and the right square commutes, since ϵ^c is a natural transformation $F_cG_c \Rightarrow I$. Hence the diagram 4.3 commutes and thus 4.2 commutes for $T = G_c(h)g_1$. For $T = g_2G_{c'}(h)$ the diagram 4.2 turns to

$$\begin{tikzcd}
F_cG_{c'}(e) \arrow[r, "F_cG_{c'}(h)"] \arrow[d, dashed, "F(f,id)"] & F_cG_{c'}(e') \arrow[r, "F_c(g_2)"] \arrow[d, dashed, "F(f,id)"] & F_cG_c(e') \arrow[d, dashed, "\epsilon_{e'}^c"] \\
F_{c'}G_{c'}(e) \arrow[r, "F_{c'}G_{c'}(h)"] \arrow[dr, "\epsilon_e^{c'}"'] & F_{c'}G_{c'}(e') \arrow[r, "\epsilon_{e'}^{c'}"] & e' \\
& e \arrow[ur, "h"'] &
\end{tikzcd}$$

The bottom triangle commutes due to the naturality of $\epsilon^{c'}$. The left square commutes since F is a bifunctor and the right square commutes by the definition of g_2. Thus the diagram 4.2 commutes for $T = g_2 G_{c'}(h)$. Hence we may define the functor $G : \mathbf{C}^{op} \times \mathbf{E} \to \mathbf{D}$ where $G(f^{op}, h) = G^{e'}(f^{op})G_{c'}(h)$ for morphisms $f : c \to c'$ and $h : e \to e'$.

It is left to see that η is natural in c. Let $f : c \to c'$ be a morphism in $\mathbf{C}$. For a reminder, fix $T : F(c,d) \to e$ and notice that $\eta_{(c,d,e)}(T)$ is the unique morphism $S : d \to G(c,e)$ where the diagram

$$\begin{tikzcd} & F_c(G_c(d)) \arrow[d, "\epsilon_e^c"] \\ F_c(d) \arrow[ur, "F_c(S)"] \arrow[r, "T"'] & e \end{tikzcd}$$

commutes. We need to see that the following square commutes:

$$\begin{tikzcd} \mathbf{E}(F(c', d)e) \arrow[r, "\eta_{(c',d,e)}"] \arrow[d, dashed, "F(f,id)^*"'] & \mathbf{D}(d, G(c', e)) \arrow[d, "G(f^{op},id)_*"] \\ \mathbf{E}(F(c,d), e) \arrow[r, "\eta_{(c,d,e)}"'] & \mathbf{D}(d, G(c,e)) \end{tikzcd}$$

Fix $T : F(c', d) \to e$. We need to see that $\eta_{(c,d,e)}(TF(f, id)) = G(f^{op}, id)\eta_{c',d,e}(T)$. In other words we need to show that the diagram

$$\begin{tikzcd} & F_cG_c(e) \arrow[d, "\epsilon_e^c"] \\ F_c(d) \arrow[ur, "F_c(g\eta'(T))"] \arrow[r, "TF(f,id_d)"'] & e \end{tikzcd} \tag{4.4}$$

commutes, where $g = G(f^{op}, id_e) : G_{c'}(e) \to G_c(e)$ and η' denotes $\eta_{c',d,e} : \mathbf{E}(F_{c'}(d), e) \to \mathbf{D}(d, G_{c'}(e))$. Consider the following diagram

$$\begin{tikzcd} F_c(d) \arrow[r, "F_c(\eta'(T))"] \arrow[dd, "F(f,id_d)"'] & F_cG_{c'}(e) \arrow[r, "F_c(g)"] \arrow[d, dashed, "F(f,id)"] & F_cG_c(e) \arrow[dd, "\epsilon_e^c"] \\ & F_{c'}G_{c'}(e) \arrow[dr, "\epsilon_e^{c'}"] & \\ F_{c'}(d) \arrow[ur, dashed, "F_{c'}(\eta'(T))"] \arrow[rr, "T"'] & & e \end{tikzcd}$$

The left side of the diagram commutes, since F is a bifunctor. The right side commutes by the definition of $g = G^e(f^{op})$ and the bottom triangle commutes since $\epsilon^{c'}$ is the unit corresponding to the natural isomorphisms $\eta_{(c',-,-)}$, where the object c' stays fixed. Thus the exterior commutes, which shows that diagram 4.4 commutes. Thus we have shown that η is a natural transformation in all three variables. □

If $F : \mathbf{C} \times \mathbf{D} \to \mathbf{E}$ is a bifunctor where F_c and F^d have right adjoints for every object c and d in their respective categories, then we have two bifunctors $G : \mathbf{C}^{op} \times \mathbf{E} \to \mathbf{D}$ and $H : \mathbf{D}^{op} \times \mathbf{E} \to \mathbf{C}$ with the following natural isomorphisms

$$\mathbf{C}(c, H(d,e)) \cong \mathbf{E}(F(c,d), e) \cong \mathbf{D}(d, G(c,e)).$$

Such a trio of functors is called a bifunctor adjoint.

Example 4.16.

1. Let $\mathbf{C}$ be a category with products and exponentials. Then we have bijections

$$\mathbf{C}(a \times b, x) \cong \mathbf{C}(a, x^b)$$

natural in a and x for every object b. The naturality follows from the fact that $(-) \times b \dashv (-)^b$. Thus by the Bifunctor Adjoint Theorem 4.15, we attain a bifunctor $[-,-] : \mathbf{C}^{op} \times \mathbf{C} \to \mathbf{C}$ which is the internal hom-functor of the category $\mathbf{C}$.

2. Let R be a commutative ring. The hom-functor $\mathrm{Hom} : \mathbf{R\text{-}mod}^{op} \times \mathbf{R\text{-}mod} \to \mathbf{Set}$ can be made **R-Mod**-valued. We ask, what is the left adjoint for this functor $\mathrm{Hom}(M, -)$. The category **R-Mod** does not have all the exponentials, since **R-mod** has a zero object. So the left adjoint cannot be a component functor of the categorical product. The left adjoint is $M \otimes (-)$, since $\mathbf{R\text{-}mod}(M \otimes N, P) \cong \mathbf{R\text{-}mod}(M, Hom(N, P))$ naturally in M and P for every N. Hence the tensor product becomes a bifunctor $\otimes : \mathbf{R\text{-}Mod} \times \mathbf{R\text{-}Mod} \to \mathbf{R\text{-}Mod}$. The tensor product is categorically similar to the categorical product functor. It has a unit object, namely the scalars R, it is associative and symmetric up to unique canonical isomorphisms that satisfy the coherence laws. This defines an example of a symmetric monoidal category.

Chapter 5

Limits and colimits

In this chapter we are going to generalize the ideas of product and sum in an arbitrary category. We are going to see how one can generalize the concepts of inverse image of a function, gluing of sets and intersection of subsets in the category **Set**.

Definition 5.1. Let $\mathbf{I}$ and $\mathbf{C}$ be categories. The functor $\Delta : \mathbf{C} \to \mathbf{C}^{\mathbf{I}}$, which is attained from the projection functor $\Pi_1 : \mathbf{C} \times \mathbf{I} \to \mathbf{C}$ via exponential transposition, is called the constant embedding functor.

The constant embedding functor $\Delta : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$, where $\mathbf{I}$ is non-empty category, is fully faithful and monic and so Δ is an embedding. Furthermore the functor Δ behaves well with respect to duality: Consider the projection functor $\Pi : \mathbf{C}^{op} \times \mathbf{I}^{op} \to \mathbf{C}^{op}$. By exponential transposition we obtain the functor $\Delta' : \mathbf{C}^{op} \to [\mathbf{I}^{op}, \mathbf{C}^{op}]$. Since the categories $[\mathbf{I}^{op}, \mathbf{C}^{op}]$ and $[\mathbf{I}, \mathbf{C}]^{op}$ are canonically isomorphic, we may identify the functors Δ^{op} and Δ'.[1]

Definition 5.2 (Limit and colimit)**.** Let $\mathbf{C}$ and $\mathbf{I}$ be categories and assume that $\mathbf{I}$ is a small category. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram and by $\overline{D} : \mathbf{1} \to [\mathbf{I}, \mathbf{C}]$ choice of the object D in $[\mathbf{I}, \mathbf{C}]$. An object of $\Delta \downarrow \overline{D}$ is called a cone over the diagram D. In the case the terminal cone exists, it is called the limit cone of the diagram D and is denoted by $\lim D$. Dually, the objects of $\overline{D} \downarrow \Delta$ are called cocones and the initial cocone is called the colimit over the diagram D and denoted colim D.

If the collection of morphisms of $\mathbf{I}$ is finite, then the corresponding (co)limit is called finite. Moreover, when we refer to the cardinality of a diagram, we refer to the cardinality of the set of morphisms of the index category. Lastly, we say that a category is (cocomplete) complete, if it contains all (colimits) limits.

Assume that $\mathbf{C}$ and $\mathbf{I}$ are categories, where $\mathbf{I}$ is a small category and $\mathbf{C}$ is a locally small category. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. The cones over $D : \mathbf{I} \to \mathbf{C}$ are pairs (c, λ), where $\lambda : \Delta(c) \Rightarrow \mathbf{D}$. The component λ_i of λ is called a leg of the cone λ for every object i in $\mathbf{I}$. The object c is called the zenith of the cone (c, λ). The object of a cocone is called nadir.

The category of cones is the category of elements $\int [\Delta(-), D]$. By Yoneda lemma, it follows that the existence of natural isomorphism $\eta_x : [x, c] \cong [\Delta(x), D]$, natural in x, is equivalent with c being the limit object of $\lim D$. The limit cone is then the universal element $\eta_c(id_c)$.

To be explicit about the duality of limits and colimits, consider the isomorphism chain:

$$\overline{D} \downarrow \Delta \cong (\Delta^{op} \downarrow \overline{D})^{op} \cong (\Delta' \downarrow \overline{D^{op}})^{op}.$$

The initial object in $(\Delta' \downarrow D^{op})^{op}$ is the same object as the terminal object in $\Delta' \downarrow D^{op}$ which is exactly the same as the limit of D^{op}. Therefore it follows that $\lim D^{op} = \text{colim } D$ and hence $\text{colim } D^{op} = \lim D$.

Example 5.3. The product of two objects c_1 and c_2 in a category $\mathbf{C}$ is the limit of the functor

[1] The square notation $[a, b]$ means the hom-set $Hom(a, b)$.

$\{1, 2\} \to \mathbf{C}$ that chooses the objects in $\mathbf{C}$:

$$
\begin{tikzcd}
 & \forall d \arrow[dl, bend right, "\forall f"'] \arrow[d, dashed, "\exists!\phi"] \arrow[dr, bend left, "\forall g"] & \\
c_1 & c_1 \times c_2 \arrow[l, "\lambda_1"] \arrow[r, "\lambda_2"'] & c_2
\end{tikzcd}
$$

Dually, the coproduct of c_1 and c_2 is the colimit of the diagram $\{1, 2\} \to \mathbf{C}$.

5.1 Products and coproducts

Before talking about the general theory of limits and colimits, we will look more deeply into some important specific limits. If a category behaves suitably nicely with, a priori, only products and equalizers, then the category has all limits.

Definition 5.4. Let $\mathbf{C}$ be a category and let I be a small set. The set I can be considered as a discrete category. The limit of a functor $F : I \to \mathbf{C}$ is called a product and the limit is denoted by

$$\prod_{i \in I} F(i).$$

The legs of the limiting cone are called projections. Dually, the colimit of F is called a coproduct and it is denoted by

$$\coprod_{i \in I} F(i).$$

The colegs, arms, of the colimiting cone are called injections.

We permit the possibility of the set I being empty and then the corresponding limit is the terminal object and the colimit is the initial object, if the corresponding universal objects exist.

Example 5.5.

1. Let L be an alphabet and let T be an algebraic L-theory. Then the category $\mathbf{Model}_L^T$ has products. Let $\mathcal{M}_i$ be an L-model satisfying the theory T for $i \in I$. If I is empty, then by Lemma 1.34 the terminal model exists. Assume then that I is non-empty. By Theorem 1.42 the product model $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$ satisfies the theory T. Furthermore the projections $p_i : \mathcal{M} :\to \mathcal{M}_i$ are L-model morphisms. As in the Example 2.54, $\mathcal{M}$ is the product object of models $\mathcal{M}_i, i \in I$.

   As a special case we attain the product of sets, monoids, groups, monoid actions and R-modules.

2. The categories of sets, monoids, groups and R-modules contain coproducts.

3. The category of small categories **Cat** contains products and coproducts.

5.2 Equalizer and coequalizer

Definition 5.6. Assume that $\mathbf{C}$ is a category and consider the diagram $a \underset{g}{\overset{f}{\rightrightarrows}} b$ in $\mathbf{C}$. If the diagram has a limit, it is called the equalizer of f. Colimit of the diagram is called a coequalizer.

Definition 5.7. Let $\mathbf{C}$ be a category with an object c. Let $S \subset \mathrm{Obj}(\mathbf{C})$. We say that the set S is jointly weakly initial, if for any object d in $\mathbf{C}$ there exists a morphism $c \to d$ for some $c \in S$. An object is called weakly initial, if the singleton it defines is a jointly weakly initial set of objects. The set of objects S is called jointly weakly terminal, if S is jointly weakly initial in $\mathbf{C}^{op}$.

Definition 5.8. Let $\mathbf{C}$ be a category. Let c and d_i be objects in $\mathbf{C}$ for every $i \in I$. Assume that $\lambda_i : c \to d_i$ is a morphism in $\mathbf{C}$ for every $i \in I$. We call the collection of morphisms $(\lambda_i)_i$ jointly monic, if $x, x' : a \to c$ holds that $f_i x = f_i x'$ for all $i \in I$ implies that $x = x'$ for all parallel pair of morphisms x and x' with codomain c. Dually, we say that a collection of morphisms is jointly epic, if it is jointly monic in the opposite category.

Theorem 5.9. *Assume that $\boldsymbol{I}$ is a category with a jointly weakly initial set of objects S. Assume that the diagram $D : \boldsymbol{I} \to \boldsymbol{C}$ has a limit (c, λ). Then λ can be identified with the morphisms $\lambda_i, i \in S$ and $(\lambda_i)_{i \in S}$ is jointly monic.*

Proof. Let k be an object in $\mathbf{I}$. There exists a morphism $j : i \to k$ for some $i \in S$ and hence $\lambda_k = D(j)\lambda_i$. So λ is completely described by $(\lambda_i)_{i\in S}$. Thus we may identify the cone λ with $(\lambda_i)_{i\in S}$. Similarly, for any cone over D.

Fix a parallel pair of morphisms $x, y : a \to c$, where $f_i x = f_i y$ for all $i \in S$. Both morphisms x and y define a cone morphism $(a, \lambda \bullet \Delta(x)) \to (c, \lambda)$ and $(a, \lambda \bullet \Delta(y)) \to (c, \lambda)$, respectively. We will show that $\lambda \bullet \Delta(x) = \lambda \bullet \Delta(y)$. Fix an object an object k in $\mathbf{I}$. Now there exists a morphism $j : i \to k$ for some $i \in S$ and so

$$\lambda_k \Delta(x)_k = \lambda_k x = D(j)\lambda_i x = D(j)\lambda_i y = \lambda_k \Delta(y)_k$$

Thus $\lambda \bullet \Delta(x) = \lambda \bullet \Delta(y)$. Because the cone morphism $(a, \lambda \bullet \Delta(x)) \to (c, \lambda)$ is unique and both morphisms x and y define such a cone morphism, it follows that $x = y$. □

Hence we see that the equalizer of a parallel pair of morphisms must be monic.

Example 5.10.

1. Consider the algebraic category $\mathbf{Model}_L^T$. By Theorem 1.23 any parallel pair of morphisms $f, g : \mathcal{M} \to \mathcal{N}$ in $\mathbf{Model}_L^T$ defines an equalizer L-model $\mathcal{A}$. Since the inclusion $\mathcal{A} \hookrightarrow \mathcal{M}$ is globally full, it follows by 1.41(3) that $\mathcal{A} \vDash T$. Because globally full injections are embeddings, the inclusion $\mathcal{A} \hookrightarrow \mathcal{M}$ is the equalizer of f and g in the category $\mathbf{Model}_L^T$.

   The categories of sets, monoids, groups and R-modules have equalizers.

2. Let L be an alphabet and let T be a positive L-theory. Then the positive category $\mathbf{Model}_L^T$ contains coequalizers: Let $\mathcal{M}$ and $\mathcal{N}$ be L,T-models with universes M and N, respectively. Consider the diagram of L,T-model morphisms

$$\begin{tikzcd} \mathcal{M} \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & \mathcal{N} \end{tikzcd}.$$

   We show the coequalizer of f and g exists. Consider the smallest L-congruence $\sim$ on $\mathcal{N}$ where $f(m) \sim g(m)$ for all $m \in M$. Denote the quotient model morphism by $q : \mathcal{N} \to \mathcal{N}/\sim$. Since the theory T is positive $\mathcal{N}/\sim$ is an L,T-model. Let $\theta : \mathcal{N} \to \mathcal{X}$ be an L-model morphism where $\theta f = \theta g$. Now $f(m) \sim_\theta g(m)$ for all $m \in M$ and so $\sim \subset \sim_\theta$. Since q is globally full surjection, θ factors through q. Hence q is the coequalizer of f and g. Thus $\mathbf{Model}_L^T$ contains coequalizers.

3. In the category $\mathbf{Cat}$ of small categories, the equalizer the diagram $\begin{tikzcd} \mathbf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] & \mathbf{D} \end{tikzcd}$ is the subcategory the domain where objects and morphisms are those that are mapped equally under F and G.

   Similarly, as with positive categories $\mathbf{Model}_L^T$, the category $\mathbf{Cat}$ of small categories has coequalizers.

5.3 Pullbacks and pushouts

The concept of a pullback generalizes the notions of intersection and the inverse image in the category $\mathbf{Set}$ of sets. Furthermore pullback is exactly a binary product in the slice category.

Pushouts, which by dually correspond to pullbacks, generalize the operation of gluing topological spaces together from suitable points, or joining of diagrams from joint arrows.

Definition 5.11. Let $\mathbf{C}$ be a category with a diagram as follows:

$$\begin{tikzcd} & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$$

The limit of the diagram is called a pullback. The limit is denote by the pullback diagram

$$\begin{tikzcd} a \times_c b \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"] \arrow[dr, phantom, "\lrcorner", very near start] & b \arrow[d, "g"] \\ a \arrow[r, "f"'] & c \end{tikzcd}$$

where the limit object is denoted $a \times_c b$. The morphism $\tilde{g}$ is called the pullback of g along f and similarly for $\tilde{f}$. If $f = g$ the morphisms $\tilde{f}$ and $\tilde{g}$ are called the kernel pair of f.

The pullback of f and g is the product of objects f and g in the slice category $\mathbf{C}/c$. We define the generalized pullback of morphisms $f_i : a_i \to c, i \in I$ as the product in the coslice category $\mathbf{C}/c$ of the objects f_i. For a fixed $i \in I$, we denote the generalized pullback diagram by

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & a_j \arrow[d, "f_j, j \neq i"] \\ a_i \arrow[r, "f_i"] & c \end{tikzcd}$$

If the morphisms $f_i, i \in I$, are monomorphisms, we call the generalized pullback an intersection of subobjects. The colimit $(a +_c b, p_1, p_2)$ of the diagram

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"] & c \\ b & \end{tikzcd}$$

is called the pushout and denoted

$$\begin{tikzcd} a \arrow[r, "g"] \arrow[d, "f"] & c \arrow[d, "p_1"] \\ b \arrow[r, "p_2"'] & b +_a c \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

The morphism p_1 is called the pushforward of the morphism f along g. Sometimes the word 'pushout' is used instead of 'pushforward'.

Example 5.12.

1. In the category **Set** of sets the pullback of a diagram

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$

is the set $X \times_Z Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$ equipped with the projections to X and Y. The pushout of

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "f"] & Z \\ Y & \end{tikzcd}$$

is the quotient set of $Y + Z$ where the congruence is generated by the requirement that $f(x)$ is identified with $g(x)$ in $Y + Z$ for all $x \in X$.

2. Define multigraphs G_1, G_2 and $\mathbf{2}$ respectively as the following multigraphs:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}$$

Let $f_i : \mathbf{2} \to G_i$ be graph homomorphisms for $i = 1, 2$, where f_1 chooses the right vertical arrow of G_1 and f_2 chooses the left vertical arrow of G_2. The obtained pushout graph is

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

Example 5.13.

1. Let $f : X \to Y$ be a function between sets and let $V \subset Y$. The following square is a pullback square in the category **Set** of sets:

$$\begin{tikzcd} f^{-1}(V) \arrow[r, "f|"] \arrow[d, hook] & V \arrow[d, hook] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

2. Let $A, B \subset X$ be sets. The following square is both a pullback and a pushout square:

$$\begin{tikzcd} A \cap B \arrow[r, hook] \arrow[d, hook] & B \arrow[d, hook] \\ A \arrow[r, hook] & A \cup B \end{tikzcd}$$

Using the idea of a pullback we can generalize the concept of an inverse image. Notice that in the category of sets we have a contravariant power set functor that takes a function f to the inverse image map f^{-1}. We can construct a functor that mirrors the contravariant power set functor.

Theorem 5.14. *Let $\boldsymbol{C}$ be a category and fix an object c in $\boldsymbol{C}$. Assume that the products $c \times a$ exist for all objects a in $\boldsymbol{C}$. Then $c \times (-)$ is a functor and moreover we have a functor $F : \boldsymbol{C} \to \boldsymbol{C}/c$, where $F(x)$ is the projection from $c \times x$ to c.*

Proof. Fix all the product objects $(c \times x, p_1, p_2)$. We define $c \times (-) : \mathbf{C} \to \mathbf{C}$ by $c \times (-)(a) = c \times a$ and $c \times (-)(f) = id_c \times f$. Via the proof of Theorem 2.56, it follows that $c \times (-)$ is a functor.

Define the functor $F : \mathbf{C} \to \mathbf{C}/c$, where $F(x)$ is the projection from $c \times x$ to c and given a morphism $f : x \to y$ in $\mathbf{C}$, we define $F(f) = id_c \times f$. Since the diagram

$$\begin{tikzcd} c \times x \arrow[rr, "id_c \times f = (pr_1, f pr_2)"] \arrow[dr, "pr_1"'] & & c \times y \arrow[dl, "pr_1"] \\ & c & \end{tikzcd}$$

commutes, it follows that F is well-defined. The functoriality of F follows from the functoriality of $c \times (-)$. □

Corollary 5.15. *Assume that $f : c \to d$ is a morphism in a category $\boldsymbol{C}$ and assume that $\boldsymbol{C}$ has all the pullbacks along f. Then there exists a canonical functor $f^{-1} : \boldsymbol{C}/d \to \boldsymbol{C}/c$ that takes a morphism to its pullback along f.*

Proof. By Theorem 5.14 we have an endofunctor $f \times' (-)$ on the category $\mathbf{C}/d$. Let $F : \mathbf{C}/d \to (\mathbf{C}/d)/f$ be a functor as defined in Theorem 5.14 that takes $g : x \to d$ to its pullback along f (the projection map from the product object to f). Since there exists a forgetful functor $(\mathbf{C}/d)/f \to (\mathbf{C}/c)$, by composition, we obtain the canonical functor $f^{-1} : \mathbf{C}/d \to \mathbf{C}/c$. □

The pullback functor along a morphism f, when exists, always restricts to a map between the subobjects:

Theorem 5.16. *Let I be a set and let the diagram*

$$\begin{tikzcd}
x \arrow[r, "g_i, i\in I"] \arrow[d, "n"'] \arrow[dr, phantom, very near start, "\lrcorner"] & y_i \arrow[d, "m_i, i\in I"] \\
a \arrow[r, "f"] & b
\end{tikzcd}$$

be a generalized pullback diagram in a category $\boldsymbol{C}$*, where the morphisms $m_i, i \in I$ are monic. Then the morphism n is monic.*

Proof. Assume that $nt = nt'$ for a parallel pair of morphisms $t, t' : c \to x$. It suffices to show that $g_i t = g_i t'$ for all $i \in I$ by Theorem 5.9. Since $m_i g_i t = fnt = fnt' = m_i g_i t'$, it follows that $gt = gt'$ by the monicness of m_i for every $i \in I$. Thus $t = t'$. $\square$

5.4 Respecting limits

In a metric space setting a function is continuous if it respects limits. We are in an analogous situation: We call a functor that preserve limits a continuous functor. There are other ways that a functor can interact with limits and diagrams. Let us first study the cone and cocone functors.

5.4.1 Cone functor

If a category $\mathbf{C}$ has a diagram D and there exists a functor $F : \mathbf{C} \to \mathbf{D}$, we attain a cone functor over the diagram D that pushes the cones over D to cones over FD. The cone functor over F inherits many important properties from the functor F.

Consider a functor $F : \mathbf{C} \to \mathbf{D}$. For clarity in the following definition we denote by $\overline{F}$ the corresponding functor $\mathbf{1} \to [\mathbf{C}, \mathbf{D}]$ via exponential transposition. Let $\mathbf{I}$ be a category. Furthermore, we denote by $\tilde{F}$ the functor $F * (-) : [\mathbf{I}, \mathbf{C}] \to [\mathbf{I}, \mathbf{D}]$.

Definition 5.17 (Cone and cocone functors)**.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and $\mathbf{D} : \mathbf{I} \to \mathbf{C}$ a diagram. Denote the terminal category by $\mathbf{1}$ and all functors with codomain $\mathbf{1}$ are denoted by !. Denote the universal diagrams of the respective comma categories by the following diagrams:

$$\begin{tikzcd}
\Delta \downarrow \overline{D} \arrow[r, "!"] \arrow[d, "P"'] & \mathbf{1} \arrow[d, "\overline{D}"] \\
\mathbf{C} \arrow[r, "\Delta"'] \arrow[ur, phantom, "\alpha"] & {[\mathbf{I}, \mathbf{C}]}
\end{tikzcd}
\quad
\begin{tikzcd}
\Delta \downarrow \overline{FD} \arrow[r, "!"] \arrow[d, "Q"'] & \mathbf{1} \arrow[d, "\overline{FD}"] \\
\mathbf{D} \arrow[r, "\Delta"'] \arrow[ur, phantom, "\beta"] & {[\mathbf{I}, \mathbf{D}]}
\end{tikzcd}
\quad
\begin{tikzcd}
\overline{D} \downarrow \Delta \arrow[r, "P'"] \arrow[d, "!"'] & \mathbf{C} \arrow[d, "\Delta"] \\
\mathbf{1} \arrow[r, "\overline{D}"'] \arrow[ur, phantom, "\gamma"] & {[\mathbf{I}, \mathbf{C}]}
\end{tikzcd}
\quad
\begin{tikzcd}
\overline{FD} \downarrow \Delta \arrow[r, "Q'"] \arrow[d, "!"'] & \mathbf{D} \arrow[d, "\Delta"] \\
\mathbf{1} \arrow[r, "\overline{FD}"'] \arrow[ur, phantom, "\delta"] & {[\mathbf{I}, \mathbf{D}]}
\end{tikzcd}$$

We define the cone and cocone functors

$$T_F : \Delta \downarrow \overline{D} \to \Delta \downarrow \overline{FD} \text{ and}$$
$$S_F : \overline{D} \downarrow \Delta \to \overline{FD} \downarrow \Delta,$$

respectively, from the following comma diagrams and using the terminality of comma categories:

$$\begin{tikzcd}
\Delta \downarrow \overline{D} \arrow[r, "!"] \arrow[d, "FP"'] & \mathbf{1} \arrow[d, "\overline{FD}"] \\
\mathbf{D} \arrow[r, "\Delta"'] \arrow[ur, phantom, "\tilde{F}\alpha"] & {[\mathbf{I}, \mathbf{D}]}
\end{tikzcd}
\quad
\begin{tikzcd}
\overline{D} \downarrow \Delta \arrow[r, "FP'"] \arrow[d, "!"'] & \mathbf{D} \arrow[d, "\Delta"] \\
\mathbf{1} \arrow[r, "\overline{FD}"'] \arrow[ur, phantom, "\tilde{F}\gamma"] & {[\mathbf{I}, \mathbf{D}]}
\end{tikzcd}$$

The functors T_F and S_F are the unique functors for which the following diagrams commute:

$$\begin{tikzcd}
\Delta \downarrow \overline{D} \arrow[dr, dashed, "T_F"] \arrow[drr, bend left, "!"] \arrow[ddr, bend right, "FP"'] & & \\
& \Delta \downarrow \overline{FD} \arrow[r, "!"] \arrow[d, dashed, "Q"] & \mathbf{1} \\
& \mathbf{D} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta FP \arrow[r, Rightarrow, bend left, "\tilde{F}\alpha"] \arrow[r, Rightarrow, bend right, "\beta * T_F"'] & \overline{FD}!
\end{tikzcd}$$

$$\begin{tikzcd}
\overline{D} \downarrow \Delta \arrow[dr, dashed, "S_F"] \arrow[drr, bend left, "FP'"] \arrow[ddr, bend right, "!"'] & & \\
& \overline{FD} \downarrow \Delta \arrow[r, "Q'"] \arrow[d, dashed, "!"] & \mathbf{D} \\
& \mathbf{1} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{FD}! \arrow[r, Rightarrow, bend left, "\tilde{F}\gamma"] \arrow[r, Rightarrow, bend right, "\delta * S_F"'] & \Delta FP'
\end{tikzcd}$$

Notice that the definitions of T_F and S_F are dual, since S_F is the same as $T^{op}_{F^{op},D^{op}}$ up to an isomorphism. Explicitly T_F and S_F are both defined as follows:

$$\begin{cases} (c,\lambda) \mapsto (Fc, F * \lambda) \\ (c,\lambda) \xrightarrow{f} (c',\lambda') \mapsto (Fc, F*\lambda) \xrightarrow{Ff} (Fc', F*\lambda'). \end{cases}$$

Definition 5.18. Let $\mathbf{C}$ and $\mathbf{D}$ be categories and let $F : \mathbf{C} \to \mathbf{D}$ be a functor. We define a functor $F' : \mathbf{D} \to \mathbf{M\text{-}CAT}$, where an object d in $\mathbf{D}$ is taken to the category $F \downarrow d$ and a morphism $g : d \to d'$ in $\mathbf{D}$ is taken to a functor $g_* : (F \downarrow d) \to (F \downarrow d')$. We define the functor g_* by

$$\begin{cases} (c, \alpha : Fc \to d) \mapsto (c, g\alpha) \\ ((c,\alpha) \xrightarrow{f} (c',\alpha')) \mapsto ((c, g\alpha) \xmapsto{f} (c', g\alpha')). \end{cases}$$

The functor F' is well-defined and we call it the induced comma category functor with respect to F.

Theorem 5.19 (Cone Functor Theorem)**.** *Let $G : \boldsymbol{D} \to \boldsymbol{C}$ be a functor and let $D : \boldsymbol{I} \to \boldsymbol{D}$ be a diagram. Then the following properties hold for the cone and cocone functors $T_{G,D} = T$ and $S_{G,D} = S$:*

1. *If G is faithful, then so are T and S.*
2. *If G is fully faithful, then so are T and S.*
3. *If G is a left adjoint, then so is S. Dually, if G is a right adjoint, then so is T.*
4. *If G is an equivalence, then so are T and S.*

Proof. By duality it suffices to prove the properties for T.

1. Assume that G is faithful. Since a cone morphism of form $f : T(d,\theta) \to T(d',\theta')$ is a morphism $f : Gd \to Gd'$, then by the faithfulness of G, it follows that T is faithful.
2. Assume that G is fully faithful. Let $g : T(c,\lambda) \to T(c',\lambda')$. We attain the following commutative diagram for every object i in I:

$$\begin{tikzcd}
Gc \arrow[r, "g"] \arrow[dr, "G(\lambda_i)"'] & Gc' \arrow[d, "G(\lambda'_i)"] \\
& Di
\end{tikzcd}$$

   Since G is fully faithful, there exists $f : c \to c'$ that defines the unique solution to the lifting problems above. Additionally, since G is faithful, the solution commutes for every $i \in I$ and so $f : (c,\lambda) \to (c',\lambda')$. Now $T(f) = g$ and thus T is fully faithful.

3. Assume that (λ, ϵ) is a unit-counit pair of adjoint $F : \mathbf{C} \dashv \mathbf{D} : G$. So $\epsilon : FG \Rightarrow I_{\mathbf{D}}$ and $\lambda : I_{\mathbf{C}} \Rightarrow GF$. We will show that T is a right adjoint. We define a left adjoint $P : \Delta \downarrow GD \to \Delta \downarrow \mathbf{D}$ for $T : \Delta \downarrow D \to \Delta \downarrow GD$ by the following commutative diagram:

$$\begin{tikzcd} \Delta \downarrow GD \arrow[r, "T'"] \arrow[dr, "P"'] & \Delta \downarrow FGD \arrow[d, "(\epsilon D)_*"] \\ & \Delta \downarrow D \end{tikzcd}$$

where $T' := T_{F,GD}$. We prove that P is a left adjoint for T by constructing a unit-counit pair. For a cone (c, η) in $\Delta \downarrow GD$ holds

$$\begin{aligned} P(c, \eta) &= (\epsilon D)_*(Fc, F\eta) \\ &= (Fc, \epsilon D \bullet F\eta). \end{aligned}$$

We define

$$\begin{aligned} &\Lambda : I \Rightarrow TP : \Delta \downarrow GD \to \Delta \downarrow GD, \Lambda_{(c,\eta)} = \lambda_c \\ &\varepsilon : PT \Rightarrow I : \Delta \downarrow D \to \Delta \downarrow D, \varepsilon_{(d,\theta)} = \epsilon_d. \end{aligned}$$

To show that the natural transformation Λ is well-defined, fix a cone (c, η) over GD. We need to see that $\lambda_c : c \to GFc$ defines a cone morphism $(c, \eta) \to (GFc, G\epsilon D \bullet GF\eta)$. Let i be an object in $\mathbf{I}$. We show that the diagram

$$\begin{tikzcd} c \arrow[r, "\lambda_c"] \arrow[d, "\eta_i"] & GFc \arrow[dl, "(G\epsilon D \bullet GF\eta)_i"] \\ D_i \end{tikzcd}$$

commutes. We see this from

$$\begin{aligned} (G\epsilon D \bullet GF\eta)_i \lambda_c &= G(\epsilon_{D_i}) GF(\eta_i) \lambda_c \\ &= G(\epsilon_{D_i}) \lambda_{G(D_i)} \eta_i \\ &= \eta_i. && \text{(unit-counit equation)} \end{aligned}$$

So λ_c is a cone morphism $(c, \eta) \to (GFc, G\epsilon D \bullet GF\eta)$. The naturality of Λ follows directly from the naturality of λ. In a similar fashion, we see that ε is well-defined natural transformation.

Next we show that the unit-counit equations

$$\begin{cases} \varepsilon P \bullet P\Lambda = P \\ T\varepsilon \bullet \Lambda T = T \end{cases}$$

hold. Let (c, η) be a cone over the diagram GD. Now

$$\begin{aligned} (\varepsilon P \bullet P\Lambda)_{(c,\eta)} &= \varepsilon_{(Fc, F\eta)} P(\Lambda_{(c,\eta)}) \\ &= \epsilon_{Fc} F(\lambda_c) \\ &= id_c \\ &= id_{(c,\eta)}. \end{aligned}$$

Similarly, the other unit-counit equation holds. Hence P is a left adjoint of T.

4. Assume that F is an equivalence and G is the inverse equivalence. We may apply the previous part to attain the unit-counit pair (Λ, ε) of $P \dashv T$. Since all the components of Λ and ε are isomorphisms, T is an equivalence.

□

5.4.2 Functors and limits

Definition 5.20 (Preservation, reflection and creation of limits and colimits)**.** Let $F : \mathbf{C} \to \mathbf{D}$ be a functor and let $\mathcal{D}$ be a collection of diagrams on $\mathbf{C}$. We say that the functor F preserves limits of the diagrams in $\mathcal{D}$, if the cone functor

$$T_F : \Delta \downarrow D \to \Delta \downarrow\ FD$$

of F preserves terminal objects for every diagram $D \in \mathcal{D}$. Dually, F preserves the colimits of $\mathcal{D}$, if the cocone functor $S_{F,D}$ preserves initial objects for every diagram $D \in \mathcal{D}$. If F preserves limits (colimits) of all small diagrams of $\mathbf{C}$, we say that F preserves all limits (colimits) and call F a continuous (cocontinuous) functor.

Similarly, we define reflection, creation and strict creation of (colimits) limits of diagrams in $\mathcal{D}$, through the reflection, creation and strict creation properties of the cone (cocone) functor with respect to the terminal (initial) objects.

Here we have an intriguing corollary of The Cone Functor Theorem.

Corollary 5.21. *Right adjoints preserve limits and left adjoints preserve colimits.*

Proof. This follows immediately from The Cone Functor Theorem 5.19 and the fact that a left adjoint functors preserve initial objects and that a right adjoint functors preserve terminal objects. □

Theorem 5.22. *Continuous functors preserve monomorphisms. Dually, cocontinuous functors preserve epimorphisms.*

Proof. It suffices to show that epimorphisms can be characterized as a colimit. Let $f : c \to d$ be a morphism. The morphism f is an epimorphism if and only if the following square is a pushout square:

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, dashed, "f"'] & b \arrow[d, dashed, "id_b"] \\ b \arrow[r, "id_b"'] & b \end{tikzcd}$$

Therefore cocontinuous functors preserve epimorphisms. □

Theorem 5.23. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a fully faithful functor. Then F reflects limits and colimits of all diagrams. If F is an equivalence, then F preserves and creates (up to an isomorphism) limits and colimits of all diagrams.*

Proof. Since F is fully faithful, it follows that T_F is fully faithful by The Cone Functor Theorem 5.19. Therefore T_F reflects all terminal objects and so F reflects all limits. Similarly, F reflects colimits.

Assume then that F is an equivalence. Thus T_F is an equivalence by the cone functor Theorem 5.19. Thus T_F preserves and creates terminal objects and thus F preserves, reflects and creates (up to an isomorphism) limits. The case regarding colimits is similar. □

Theorem 5.24. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. Then the following assertions hold:*

1. *If F reflects limits, then F reflects isomorphisms and monomorphisms.*
2. *If $\boldsymbol{C}$ is complete, F preserves limits and if F reflects isomorphisms, then F reflects limits.*

Proof.

1. Assume that F reflects limits and let $f : a \to b$ be a morphism in $\mathbf{C}$. The morphisms f is monic if and only if the pair of morphisms (f, f) is a kernel pair for id_b. Thus F reflects monomorphisms. The morphisms f is an isomorphisms if and only if it is an equalizer of a pair of identities on b. Thus F reflects isomorphisms.

2. Assume that $\mathbf{C}$ is complete, F preserves limits and assume that F reflects isomorphisms. Assume that $D : \mathbf{I} \to \mathbf{C}$ is a diagram with a cone (c, λ) where $(Fc, F\lambda)$ is a limit cone over FD. We need to show that (c, λ) is a limit cone over D. Because $\mathbf{C}$ is complete, it follows that there exists a limit cone (c', λ') over D. Thus there exists a cone morphisms $(c, \lambda) \xrightarrow{f} (c', \lambda')$. Since F preserves limits, it follows that Ff is an isomorphisms. Because F reflects isomorphisms, f is an isomorphism and thus (c, λ) is a limit cone over D.

□

Theorem 5.25. *Let $\boldsymbol{C}$ be a locally small category and let x be an object in $\boldsymbol{C}$. Then the hom-functors $\boldsymbol{C}_x$ and $\boldsymbol{C}^x$ preserve limits and colimits, respectively.*

Proof. It suffices to show that $\mathbf{C}_x$ preserves limits, since the preservation of colimits by $\mathbf{C}^x$ is formally dual. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram in $\mathbf{C}$.

Assume that (c, λ) is a cone over the diagram D. We show that $(\mathbf{C}(x, c), \mathbf{C}_x * \lambda)$ is terminal in the category of cones $\Delta \downarrow (\mathbf{C}_x \circ D)$. Let (A, η) be any cone over $\mathbf{C}_x \circ D$. We show that there exists a unique cone morphism $(A, \eta) \xrightarrow{f} (\mathbf{C}(x, c), \mathbf{C}_x * \lambda)$. Notice that $\eta(a) := (\eta_i(a) : x \to D_i)_i$ is a cone over D, because given $f : i \to j$ in $\mathbf{I}$, we have $Df \circ \eta_i(a) = (\mathbf{C}_x(Df) \circ \eta_i)(a) = \eta_j(a)$.

Uniqueness: Assume that such a cone morphism $f : A \to \mathbf{C}(x, c)$ exists. Denote $f_a := f(a) : x \to c$. To see the uniqueness, it suffices to show that f_a is a cone morphism $(x, \eta(a)) \to (c, \lambda)$. We need to show that $\lambda_i f(a) = \eta_i(a)$ for all objects i in $\mathbf{I}$ and $a \in A$. This follows directly from the fact that $(\mathbf{C}_x * \lambda) \bullet \Delta(f) = \eta$.

Existence: Define a morphism f_a as the unique cone morphism $(x, \eta(a)) \to (c, \lambda)$ for every $a \in A$ and we obtain $f : A \to \mathbf{C}(x, c)$. It remains to show that $(\mathbf{C}_x * \lambda) \bullet \Delta(f) = \eta$. Since the equality holds pointwise, it holds. Thus f is a cone morphism $(A, \eta) \to (\mathbf{C}(x, c), \mathbf{C}_x * \lambda)$. Therefore $\mathbf{C}_x$ preserves limits. □

Theorem 5.26. *Let F and G be functors $\boldsymbol{C} \to \boldsymbol{D}$. Assume that there exists an isomorphism $\eta : F \cong G : \boldsymbol{C} \to \boldsymbol{D}$. Then F and G preserve, reflect and create (up to an isomorphism) the same limits and colimits isomorphically.*

Proof. It suffices, by duality, to check only the cases regarding limits. Assume that $D : \mathbf{I} \to \mathbf{D}$ is a diagram. We have the cone functors $T_F : \Delta_{\mathbf{C}} \downarrow D \to \Delta_{\mathbf{D}} \downarrow FD$ and $T_G : \Delta_{\mathbf{C}} \downarrow D \to \Delta_{\mathbf{D}} \downarrow GD$. Since $\eta : F \cong G : \mathbf{C} \to \mathbf{D}$, we obtain $\eta * D : FD \cong GD$. Consider the following diagram

$$\begin{tikzcd} \Delta_{\mathbf{C}} \downarrow D \arrow[r, "T_F"] \arrow[dr, "T_G"'] & \Delta_{\mathbf{D}} \downarrow FD \arrow[d, "(\eta D)_*"] \\ & \Delta_{\mathbf{D}} \downarrow GD \end{tikzcd}$$

and notice that the horizontal functor is an isomorphism of categories. Therefore, if we show that the functors T_G an $(\eta D)_* T_F$ are isomorphic via a natural isomorphism, then by Theorem 3.25, the claim follows. Now

$$\begin{aligned} (\eta D)_* T_F(c, \lambda) &= (\eta D)_*(Fc, F\lambda) \\ &= (Fc, \eta D \bullet F\lambda) \\ &= (Fc, G\lambda \bullet \Delta(\eta_c)). \end{aligned}$$

The last equation uses the naturality of η. Define a natural transformation

$$\theta : (\eta D)_* T_F \Rightarrow T_G$$

by setting $\theta_{(c,\lambda)} = \eta_c : (Fc, G\lambda \bullet \Delta(\eta_c)) \to (Gc, G\lambda)$.

Clearly η_c is a cone morphism. The naturality of θ follows directly from the naturality of η. Since every component of θ is an isomorphism, it follows that θ is an isomorphism, which proves the claim.

□

Corollary 5.27. *Every covariant representable functor preserves limits and every contravariant representable functor preserves colimits.*

Proof. The claim follows as a direct application of Theorem 5.25 and Theorem 5.26. □

5.5 Completeness

As previously defined, on one hand a category is complete if it is closed under limit operations. Especially, it must have all products and equalizers. This, on the other hand, characterizes completeness. Before proving the general case, let's look at the situation in the category **Set** of sets.

Theorem 5.28. *The category **Set** of sets is complete.*

Proof. Let $D : \mathbf{I} \to \mathbf{Set}$ be a diagram on the category of sets. We will show that the diagram D has a limit. By Yoneda lemma, it suffices to see that there exists a set X with an isomorphism

$$Hom(A, X) \cong Hom(\Delta(A), D).$$

natural in A. Let A be a set and by 1 we denote a singleton set. Assuming that the limit object X exists, it becomes a representation for the contravariant functor $\mathrm{Hom}(-, D) \circ \Delta^{op}$. Therefore we would have the following chain of bijections

$$\begin{aligned} X &\cong \mathrm{Hom}(1, X) \\ &\cong \mathrm{Hom}(\Delta_1, D) \\ &\cong \left\{ (\lambda_i)_i \in \prod_{i \in \mathrm{Obj}(\mathbf{I})} D_i \mid (Df(\lambda_{\mathrm{dom}(f)}))_{f \in \mathrm{Mor}(\mathbf{I})} = (\lambda_{cod(f)})_{f \in \mathrm{Mor}(\mathbf{I})} \right\}. \end{aligned}$$

So we have a guess what the limit of D looks like. We will define explicitly

$$\lim D := \left\{ (\lambda_i)_i \in \prod_{i \in \mathrm{Obj}(\mathbf{I})} D_i \mid (Df(\lambda_{\mathrm{dom}(f)}))_{f \in \mathrm{Mor}(\mathbf{I})} = (\lambda_{cod(f)})_{f \in \mathrm{Mor}(\mathbf{I})} \right\}$$

with the associated restriction maps $p_j : \lim D \to D_j$ of the projections for objects j in $\mathbf{I}$.

Notice that an other way to define the same set $\lim D$ would be to take the equalizer of the following morphisms

$$\begin{aligned} s, t &: \prod_i D_i \to \prod_f D(\mathrm{cod}\ (f)), \text{ where} \\ s(\lambda) &= (Df(\lambda_{\mathrm{dom}(f)}))_{f \in \mathrm{Mor}(\mathbf{I})} \text{ and} \\ t(\lambda) &= (\lambda_{cod(f)})_{f \in \mathrm{Mor}(\mathbf{I})}. \end{aligned}$$

The maps s and t can be also defined from the following diagram

$$\begin{tikzcd} & D(\mathrm{dom}(f)) \arrow[r, "Df"] & D(\mathrm{cod}(f)) \\ E \arrow[r, "j"] & \prod_i Di \arrow[u, dashed, "pr_{\mathrm{dom}(f)}"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[dr, bend right, "pr_{\mathrm{cod}(f)}"'] & \prod_f D(\mathrm{cod}(f)) \arrow[u, dashed, "pr_f"'] \arrow[d, dashed, "pr_f"] \\ & & D(\mathrm{cod}(f)) \end{tikzcd}$$

such that the top and bottom halves of the diagram commute for every morphism f in $\mathbf{I}$. The pair (E, j) denotes the equalizer of s and t. We will show that this categorical construction yields the limit object. Therefore the proof generalizes from the specific category **Set**.

The collection of maps $\eta_i = pr_i \circ j : E \to \prod_i Di \to Di$ becomes a cone over D: Let $f : i \to i'$ be a morphism in $\mathbf{I}$. We need to see that the diagram

$$\begin{tikzcd} E \arrow[r, "\eta_i"] \arrow[dr, "\eta_{i'}"'] & Di \arrow[d, dashed, "Df"] \\ & Di' \end{tikzcd}$$

commutes. Notice that

$$\begin{aligned} Df \circ \eta_i &= Df \circ pr_i \circ j \\ &= pr_f \circ s \circ j \\ &= pr_f \circ t \circ j \\ &= pr_{i'} j \\ &= \eta_{i'}. \end{aligned}$$

Therefore η is a cone over D. It remains to be shown that η, in fact, is a terminal cone. Let (X, θ) be any other cone over D. We need to show that there exists a unique morphism $g : X \to E$ that factors the legs of θ through the corresponding legs of η:

$$\begin{tikzcd} \forall X \arrow[r, dotted, "\exists! g"] \arrow[rr, bend right, "\forall \theta_i"'] & E \arrow[r, "\eta_i"] & D_i \end{tikzcd}$$

Since θ is a cone over D, by the universal property of products, θ defines a morphism $X \to \prod_i D_i$, which we shall also denote by θ. Now given any morphism $f : i \to i'$ in $\mathbf{I}$, we have

$$\begin{aligned} pr_f s\theta &= Df \circ pr_i \theta \\ &= Df \circ \theta_i \\ &= \theta_{i'} \\ &= pr_{i'} \theta \\ &= pr_f \circ t \circ \theta. \end{aligned}$$

Thus by the universal property of products, $t \circ \theta = s \circ \theta$ and hence by the fact that (E, j) is the equalizer of s and t there exists a unique morphism $g : X \to E$ where

$$j \circ g = \theta.$$

This is equivalent to saying that they equal in the components. Thus $\eta_i \circ g = \theta_i$ for all objects i in $\mathbf{I}$. This shows that (E, η) is the limit cone of the diagram D. □

Since the proof of Theorem 5.28 was done categorically only using products and equalizers we get an immediate corollary:

Corollary 5.29. *Let $\boldsymbol{C}$ be a category and let κ be a cardinal. Assume that $\boldsymbol{C}$ has equalizers and all products of a size at most κ. Then all diagrams on $\boldsymbol{C}$ of the size at most κ have a limit. Especially, if $\boldsymbol{C}$ contains all equalizers and all products, then $\boldsymbol{C}$ is complete.*

Corollary 5.30. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a functor. Assume that $\boldsymbol{C}$ is complete. If F preserves products and equalizers, then F preserves all limits.*

Proof. Let $D : \mathbf{I} \to \mathbf{C}$ be a diagram. Consider the diagram

$$\begin{tikzcd} & D(\mathrm{dom}(f)) \arrow[r, "Df"] & D(\mathrm{cod}(f)) \\ E \arrow[r, "k"] & \prod_i Di \arrow[u, "pr_{\mathrm{dom}(f)}"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[dr, bend right, "pr_{\mathrm{cod}(f)}"'] & \prod_f D(\mathrm{cod}(f)) \arrow[u, "pr_f"'] \arrow[d, "pr_f"] \\ & & D(\mathrm{cod}(f)) \end{tikzcd}$$

where the upper and lower parts commute for all morphisms f in $\mathbf{I}$ and k is the equalizer of s and t. The limit of D is defined by the morphisms $pr_i k$ for objects i in $\mathbf{I}$. By applying the functor F on the diagram above and the fact that F preserves products and equalizers, it follows that $F(E)$ with a the cone $(F(pr_i)F(k))_i$ defines the limit diagram over FD. We see the terminality from the proof of Theorem 5.28. Since F preserves this particular limit cone, it follows preserves all limit cones. □

These results dualize; a category with coproducts and coequalizers has all colimits. A functor from a cocomplete category preserves colimits, if it preserves coproducts and coequalizers.

If a category has a terminal object, binary products and equalizers, then it has all finite limits. The existence of finite products is seen, for example, by an inductive argument using binary products.

Corollary 5.31. *All algebraic categories $\boldsymbol{Model}_L^T$ are complete, where L is an alphabet and T is an algebraic L-theory. Moreover, every positive category $\boldsymbol{Model}_{L'}^{T'}$ that has coproducts is cocomplete.*

Proof. Algebraic categories have products and equalizers by Example 2.54(1) and Example 5.10(1). Positive categories have coequalizers by Example 5.10(2). □

The category of sets, monoids, R-modules and small categories are complete and cocomplete categories. The following theorem shows that the category of topological spaces is complete and cocomplete.

Theorem 5.32. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a faithful functor and let $D : \boldsymbol{I} \to \boldsymbol{C}$ be a diagram. Assume that $\lambda_i : c \to Di$ is a morphism in $\boldsymbol{C}$ for every object i of $\boldsymbol{I}$. Assume that $(Fc, (F\lambda_i)_i)$ is the limit of FD. Then the collection of morphisms $\lambda = (\lambda_i)_i$ F-induces the structure on c if and only if (c, λ) is the limit of D.*

Proof. Since F is faithful and $(F\lambda_i)_i$ is a natural transformation, it follows that $\lambda = (\lambda_i)_i$ is a natural transformation $\Delta(c') \Rightarrow D$. Assume that the collection of morphisms λ induces the structure on c. We show that (c, λ) is the limit over D. Assume that (c', η) is cone over D. Now $(Fc', F\eta)$ is a cone over FD. Since $(Fc, F\lambda)$ is a limit cone, we have a unique cone morphism $g : (Fc', F\eta) \to (Fc, F\lambda)$. In other words the diagram

$$\begin{tikzcd} Fc' \arrow[r, "g"] \arrow[dr, "F\eta_i"'] & Fc \arrow[d, dashed, "F\lambda_i"] \\ & FD_i \end{tikzcd}$$

commutes for every object i in $\mathbf{I}$. Since the morphisms λ_i induce the structure on c, it follows that there exists a morphism $f : c' \to c$ where $Ff = g$. By the faithfulness of F, f is a unique cone morphism $(c', \eta) \to (c, \lambda)$. Thus (c, λ) is the limit of D.

Assume then that $\lambda : \Delta(c) \Rightarrow D$ is a limit cone over D. We show that λ induces the structure on c. Assume that the diagram

$$\begin{tikzcd} Fc' \arrow[r, "h"] \arrow[dr, "F\eta_i"'] & Fc \arrow[d, dashed, "F\lambda_i"] \\ & FD_i \end{tikzcd}$$

commutes, where $\eta_i : c' \to Di$ is a morphism for every object i in $\mathbf{I}$. Notice that $\eta = (\eta_i)_i$ is a natural transformation $\Delta(c') \to D$ due to the faithfulness of F and the fact that $F\lambda \bullet \Delta(h)$ is natural. Because λ is a limit cone, it follows that there exists a cone morphism $f : (c', \eta) \to (c, \lambda)$. Since $Ff, g : (Fc', F\eta) \to (Fc, \lambda)$ are cone morhpisms, it follows by the terminality of (Fc, λ) that $Ff = g$. Thus λ induces the structure on c. □

Corollary 5.33. *Let $F : \boldsymbol{C} \to \boldsymbol{D}$ be a topological functor. Then F creates limits and colimits. Especially if $\boldsymbol{D}$ is complete or cocomplete, so is $\boldsymbol{C}$.*

Proof. If $D : \mathbf{I} \to \mathbf{C}$ is a diagram and FD has a limit (d, λ), then by Theorem 5.32 the F-induced structure (c, θ) of (d, λ) is a limit cone over D. □

We see that the category of topological spaces is complete and cocomplete and the limits are obtained from corresponding limits in **Set** where the legs of the limit cones induce the the topology on the limit set. Dually the arms of a colimit cocones coinduce the topology on the colimit set. A similar statement is true for the category of measurable spaces. Perhaps not so surprisingly a proset category with products is topological over the terminal category **1**. Thus a proset with arbitrary infimums is complete.

Before ending the conversation about limits and completeness, we should see how completeness follows through some constructions. In the theory of metric spaces, many function spaces with a complete codomain, become complete themselves. This idea analogously transfers to the exponential categories, which are also called functor categories.

Theorem 5.34. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. Let $D : \boldsymbol{I} \to [\boldsymbol{C}, \boldsymbol{D}]$ be a diagram. If the limit of diagram $Ev_c \circ D : \boldsymbol{I} \to \boldsymbol{D}$ exists for all objects c in $\boldsymbol{C}$, then the diagram D has a limit which can be computed pointwise. Especially, if $\boldsymbol{D}$ is complete, then so is $[\boldsymbol{C}, \boldsymbol{D}]$.*

Proof. To simplify the notation, we will use the same symbol "D" to denote all the possible variations of the same information of D as a functor $\mathbf{I} \to [\mathbf{C}, \mathbf{D}], \mathbf{I} \times \mathbf{C} \to \mathbf{D}, \mathbf{C} \to [\mathbf{I}, \mathbf{D}]$. Fix the limits of $Dc = Ev_c \circ D$ and denote the natural transformations associated to the cones by η^c. Define a functor $F : \mathbf{C} \to \mathbf{D}$ where $Fc = \lim D_c$ and for a morphism $f : c \to c'$ define $Ff : Fc \to Fc'$ to be a unique cone morphism making the diagram

$$\begin{tikzcd}
\lim Dc \arrow[r, dotted, "Ff"] \arrow[d, "\eta_i^c"] & \lim Dc' \arrow[d, "\eta_i^{c'}"] \\
D(c,i) \arrow[r, "D(f, id_i)"'] & D(c',i)
\end{tikzcd}$$

commute for every object i in $\mathbf{I}$. By the usual uniqueness argument, we see that F is a functor. It remains to be seen that F is the limit of the diagram $D : \mathbf{I} \to [\mathbf{C}, \mathbf{D}]$. To see that F becomes a cone over D we need to define a natural transformation $\lambda_i : F \Rightarrow Di$ for all objects i in $\mathbf{I}$. This is done by choosing $\lambda_{(i,c)} = \eta_i^c$. The naturality of λ_i is exactly the statement that the above diagram commutes for an object i in $\mathbf{I}$.

We need to see that the collection $\lambda = (\lambda_i)_i$ itself becomes a cone over D and moreover a universal cone. Given any morphism $k : i \to j$ in $\mathbf{I}$, we need to verify that the diagram

$$\begin{tikzcd}
F \arrow[r, "\lambda_i"] \arrow[dr, "\lambda_j"'] & Di \arrow[d, "Dk"] \\
& Dj
\end{tikzcd}$$

commutes. So it suffices to check that

$$\begin{tikzcd}
Fc \arrow[r, "\lambda_{(i,c)}"] \arrow[dr, "\lambda_{(j,c)}"'] & D(i,c) \arrow[d, "D(k, id_c)"] \\
& D(j,c)
\end{tikzcd}$$

commutes for all objects c in $\mathbf{C}$. This follows directly from the fact that the pair (Fc, η^c) is a cone over Dc.

Lastly we need to check the terminality of our cone (F, λ). Assume that (G, θ) is a cone over the diagram $D : \mathbf{I} \to [\mathbf{C}, \mathbf{D}]$. By the naturality of θ, (Gc, θ_c) is a cone over Dc for all objects c in $\mathbf{C}$. Therefore there exists a unique morphism $g_c : Gc \to Fc$ that factors the legs of θ_c through the legs of η_c: The diagram

$$\begin{tikzcd}
Gc \arrow[d, dotted, "g_c"] \arrow[dd, bend right, "\theta_{(c,i)}"'] \\
Fc \arrow[d, "\eta_i^c"] \\
D(i,c)
\end{tikzcd}$$

commutes for every i in $\mathbf{I}$. This shows the uniqueness of the cone morphism $(G, \theta) \to (F, \lambda)$.

For existence we will check that $g = (g_c)_c$ becomes a natural transformation $G \Rightarrow F$ and this finishes the proof. Fix a morphism $f : c \to c'$ in $\mathbf{C}$. We need to show that the diagram

$$\begin{tikzcd}
Gc \arrow[r, "g_c"] \arrow[d, "Gf"] & Fc \arrow[d, "Ff"] \\
Gc' \arrow[r, "g_{c'}"'] & Fc'
\end{tikzcd}$$

commutes. Since $(Fc', \eta^{c'})$ is a cone over Dc, it suffices to show that $\eta_i^{c'} \circ (Ff \circ g_c) = \eta_i^{c'} \circ (g_{c'} \circ Gf)$

for all objects i in $\mathbf{I}$. Notice that from the diagram

$$\begin{tikzcd} Gc \arrow[r, "g_c"] \arrow[d, "Gf"] \arrow[rr, bend left, "\theta_{(i,c)}"] & Fc \arrow[r, "\eta_i^c"] \arrow[d, "Ff"] & D(i,c) \arrow[d, "D(i,f)"] \\ Gc' \arrow[r, "g_{c'}"'] \arrow[rr, bend right, "\theta_{(i,c')}"'] & Fc' \arrow[r, "\eta_i^{c'}"'] & D(i,c') \end{tikzcd}$$

all but the left square are known to commute for all objects i in $\mathbf{I}$. Because $(Fc', \eta^{c'})$ is a limit cone over $D(c')$ and $\eta_i^{c'} F(f) g_c = \eta_i^{c'} g_{c'} Gf$ for all objects i in $\mathbf{I}$, then by the uniqueness of the cone morphism $Gc \to Fc'$ it follows that $F(f)g_c = g_{c'}Gf$. Thus g is a natural transformation and the cone (F, λ) is a terminal cone. □

Theorem 5.34 combined with Yoneda lemma says that any locally small category can be embedded into a complete and cocomplete category.

Corollary 5.35. *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be categories. The constant embedding functor $\Delta : \boldsymbol{D} \to [\boldsymbol{C}, \boldsymbol{D}]$ is a continuous functor and dually a cocontinuous functor.*

Proof. Let $D : \mathbf{I} \to \mathbf{D}$ be a diagram in $\mathbf{D}$ with a limit cone (d, η) over D. We will show that $(\Delta(d), \Delta * \eta)$ is a limit cone over $\Delta \circ D$. By the proof of Theorem 5.34, the limit of $\Delta \circ D$ exists, since the limit of $Ev_c \circ \Delta \circ D = D$ exists for all objects c in $\mathbf{C}$. The limit functor $F : \mathbf{C} \to \mathbf{D}$ can be chosen to be the constant functor $\Delta(d)$. We hence have a limit cone (F, λ), where $\lambda_c = \eta$ for all objects c in $\mathbf{C}$, which shows that $(F, \lambda) = (\Delta(d), \Delta * \eta)$.

The cocontinuity of Δ follows by considering $\Delta^{op} \cong \Delta'$ via the isomorphism $[\mathbf{C}^{op}, \mathbf{D}^{op}] \cong [\mathbf{C}, \mathbf{D}]^{op}$ where $\Delta' : \mathbf{D}^{op} \to [\mathbf{C}^{op}, \mathbf{D}^{op}]$. [2] □

Theorem 5.36. *Let $\boldsymbol{C} \xrightarrow{F} \boldsymbol{E} \xleftarrow{G} \boldsymbol{D}$ be functors. Let the following diagram be the universal diagram of comma category $F \downarrow G$:*

$$\begin{tikzcd} F \downarrow G \arrow[r, "R"] \arrow[d, "L"] & \boldsymbol{D} \arrow[d, "G"] \\ \boldsymbol{C} \arrow[r, "F"'] \arrow[ur, phantom, "\Uparrow\gamma"] & \boldsymbol{E} \end{tikzcd}$$

Fix a diagram $D : \boldsymbol{I} \to F \downarrow G$. Assume that limits of the diagrams LD and RD exist and that G preserves the limit of RD. Then the limit of D exists and the forgetful functors L and R preserve it.

Proof. The diagrams LD and RD have limits (s, η) and (t, θ), respectively. To define an object in $F \downarrow G$, we need a morphism $f : Fs \to Gt$. Since G preserves the limit over RD, $(Gt, G\theta)$ is a limit over the diagram GD in $\mathbf{E}$. Thus we may define a morphism f as the unique morphism that makes the diagram

$$\begin{tikzcd} F(s) \arrow[r, "f"] \arrow[d, "F\eta_i"] & G(t) \arrow[d, "G\theta_i"] \\ FLDi \arrow[r, "\gamma_{Di}"'] & GRDi \end{tikzcd}$$

commute for all objects i in $\mathbf{I}$. The morphism $f : Fs \to Ft$ is well-defined since $\gamma D \bullet F\eta$ is a cone over GRD.

Denote the collection $(\eta_i, \theta_i)_i$ by (η, θ). By the definition of f, it follows that, (η_i, θ_i) is a morphism $(s, t, f) \to Di$ in $F \downarrow G$ for all objects i in $\mathbf{I}$. We will show that (η, θ) defines a limit cone over D. First we will show that (η, θ) is a cone. Fix a morphism $k : i \to j$. We will need to show that the diagram

$$\begin{tikzcd} (s,t,f) \arrow[r, "(\eta_i,\theta_i)"] \arrow[dr, "(\eta_j,\theta_j)"'] & Di \arrow[d, "Dk"] \\ & Dj \end{tikzcd}$$

[2]The isomorphism $\Delta^{op} \cong \Delta'$ is considered in the slice category, not via a natural isomorphism.

commutes. Now

$$\begin{aligned} Dk \circ (\eta_i, \theta_i) &= (LDk, RDk) \circ (\eta_i, \theta_i) \\ &= (LDk \circ \eta_i, RDk \circ \theta_i) \\ &= (\eta_j, \theta_j). \end{aligned}$$

The last equation holds by the fact that η and θ are cones over LD and RD, respectively. Notice that a collection of pairs of morphisms $(\alpha_i, \beta_i)_i$ is a cone over D if and only if α and β are cones over LD and RD, respectively.

To show that (η, θ) defines a terminal cone, fix a cone $(x, (\alpha, \beta))$ over D. Thus α and β are cones over LD and RD. We will show that there exists a unique cone morphism $(x, (\alpha, \beta)) \to ((s, t, f), (\eta, \theta))$.

Uniqueness: Let $(p_1, p_2) : (x, (\alpha, \beta)) \to ((s, t, f), (\eta, \theta))$ be a cone morphism. Now

$$(\eta_i, \theta_i) \circ (p_1, p_2) = (\eta_i p_1, \theta_i p_2) = (\alpha_i, \beta_i), \ i \text{ in } \mathbf{I}.$$

Thus p_1 and p_2 become cone morphisms $(x_1, \alpha) \to (s, \eta)$ and $(x_2, \beta) \to (t, \theta)$ which shows the uniqueness.

Existence: There exist cone morphisms $p_1 : (x_1, \alpha) \to (s, \eta)$ and $p_2 : (x_2, \beta) \to (t, \theta)$. If $(p_1, p_2) : x \to (s, t, f)$, then the pair (p_1, p_2) defines the wanted cone morphism. Thus it suffices to show that the diagram

$$\begin{tikzcd} Fx_1 \arrow[r, "x_3"] \arrow[d, dashed, "Fp_1"] & Gx_2 \arrow[d, dashed, "Gp_2"] \\ Fs \arrow[r, "f"'] & Gt \end{tikzcd}$$

commutes. Here we use the fact that $(Gt, G * \theta)$ is the terminal cone over GRD. Notice that the diagram

$$\begin{tikzcd} Fx_1 \arrow[r, "x_3"] \arrow[d, dashed, "Fp_1"] \arrow[dd, bend right, "F\alpha_i"'] & Gx_2 \arrow[d, dashed, "Gp_2"] \arrow[dd, bend left, "G\beta_i"] \\ Fs \arrow[r, "f"'] \arrow[d, dashed, "F\eta_i"] & Gt \arrow[d, dashed, "G\theta_i"] \\ FLDi \arrow[r, "\gamma_{Di}"'] & GRDi \end{tikzcd}$$

is known to commute outside of the top square for all objects i in $\mathbf{I}$. Therefore

$$\begin{aligned} G\theta_i \circ (Gp_2 \circ x_3) &= G\beta_i x_3 \\ &= \gamma_{Di} F\alpha_i \\ &= \gamma_{Di} F\eta_i Fp_1 \\ &= G\theta_i \circ (f \circ Fp_1), \end{aligned}$$

for all i in $\mathbf{I}$. Hence $Gp_2 \circ x_3 = f \circ Fp_1$ and so (p_1, p_2) becomes a morphism $x \to (s, t, f)$ in $F \downarrow G$.

Directly from construction, we see that the forgetful functors L and R preserve the limit of D (and hence any limit cone over D). □

Corollary 5.37 (Completeness of a comma category)**.** *Let $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ be categories and let $\boldsymbol{C} \xrightarrow{F} \boldsymbol{E} \xleftarrow{G} \boldsymbol{D}$ be functors. Assume that G is continuous and the categories $\boldsymbol{C}$ and $\boldsymbol{D}$ are complete. Then the comma category $F \downarrow G$ is complete and the forgetful functors $L : F \downarrow G \to \boldsymbol{C}$ and $R : F \downarrow G \to \boldsymbol{D}$ preserve and create limits. Dually, if F is cocontinuous and $\boldsymbol{C}$ and $\boldsymbol{D}$ are cocomplete, then $F \downarrow G$ is cocomplete and the same forgetful functors preserve and create colimits.*

Proof. The completeness is immediate from Theorem 5.36. From the completeness of $\mathbf{C}$ and $\mathbf{D}$ and the continuity of G, it follows that the forgetful functors preserve and create limits in $F \downarrow G$. The duality is seen by the fact that $G^{op} \downarrow F^{op} \cong (F \downarrow G)^{op}$ and F is continuous if and only if F^{op} is cocontinuous. □

Example 5.38.

1. If $\mathbf{C}$ is a complete category, then the coslice category $c/\mathbf{C}$ is complete. This is seen by the fact that $c/\mathbf{C} \cong c \downarrow I_{\mathbf{C}}$
   (a) The category of pointed sets $\mathbf{Set}_* \cong \{*\}/\mathbf{Set}$ is complete.
   (b) Similarly, the category of pointed spaces $\mathbf{Top}_*$ is complete.
2. Dually, the slice category $\mathbf{C}/c \cong \mathbf{C} \downarrow c$ is cocomplete if the category $\mathbf{C}$ is cocomplete.
3. If a category $\mathbf{C}$ is complete or cocomplete, so is the arrow category $\mathrm{Ar}(\mathbf{C}) \cong \mathbf{C} \downarrow \mathbf{C}$.
4. If $\mathbf{C}$ is a complete or a cocomplete category and $D : \mathbf{I} \to \mathbf{C}$ is a diagram in $\mathbf{C}$, then the cocone or cone category over D is complete or cocomplete, respectively. This is true, since the cocone category is $D \downarrow \Delta$, the cone category is $\Delta \downarrow D$ and the functor $\Delta : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$ is both continuous and cocontinuous.
5. Let R be a commutative ring. A commutative unital algebra A over R consists of a commutative ring structure and a ring homomorphism $f : R \to A$, called an R-scalar multiplication. The morphisms between such algebras A are exactly those ring homomorphisms, that respect the choice of scalar multiplication. Hence the category of commutative unital algebras over R is isomorphic to $R \downarrow \mathbf{CRing}$. Since the category $\mathbf{CRing}$ is complete, so is the category of commutative unital algebras over the ring R.

Define the forgetful functor U from $\mathbf{C}/c$ to $\mathbf{C}$ be the functor that maps objects to their domains and morphisms in $\mathbf{C}/c$ to themselves in $\mathbf{C}$.

Theorem 5.39. *Let $\mathbf{C}$ be a complete category with an object c. Then the slice category $\mathbf{C}/c$ is complete.*

Proof. We will show that the slice category has products and equalizers and this shows that $\mathbf{C}/c$ is complete. We will start with the product, so fix a collection of morphism $f_i : a_i \to c$ in $\mathbf{C}$ where i runs through an index set I and denote the diagram in $\mathbf{C}/c$ by $D : I \to \mathbf{C}/c$. We identify the set I with the discrete set of objects is I. If the index set I is empty, then the limit is the terminal object id_c in $\mathbf{C}/c$. We may assume that I is non-empty. Hence we may denote a diagram $D : \mathbf{J} \to \mathbf{C}$ in $\mathbf{C}$. Here $\mathbf{J}$ is the category with objects $I \sqcup \{*\}$ where $*$ is defined to be the terminal object of $\mathbf{J}$. Furthermore we allow no other non-trivial morphisms $\mathbf{J}$ to exist. Additionally, we define $D(i \to *) = f_i$. So the diagram D looks as follows:

$$\begin{tikzcd}
a_1 \arrow[drr, "f_1"'] & a_2 \arrow[dr, "f_2"'] & a_3 \arrow[d, "f_3"] & a_4 \arrow[dl, "f_4"] & a_5 \arrow[dll, "f_5"] \\
& & c & &
\end{tikzcd}$$

The limit of D exists in $\mathbf{C}$. Therefore there exists a limit cone (a, η) over D:

$$\begin{tikzcd}
& & a \arrow[dll, "\eta_1"'] \arrow[dl, "\eta_2"'] \arrow[d, "\eta_3"] \arrow[dr, "\eta_4"] \arrow[drr, "\eta_5"] \arrow[dd, bend left, "\eta_*"] & & \\
a_1 \arrow[drr, "f_1"'] & a_2 \arrow[dr, "f_2"'] & a_3 \arrow[d, "f_3"] & a_4 \arrow[dl, "f_4"] & a_5 \arrow[dll, "f_5"] \\
& & c & &
\end{tikzcd}$$

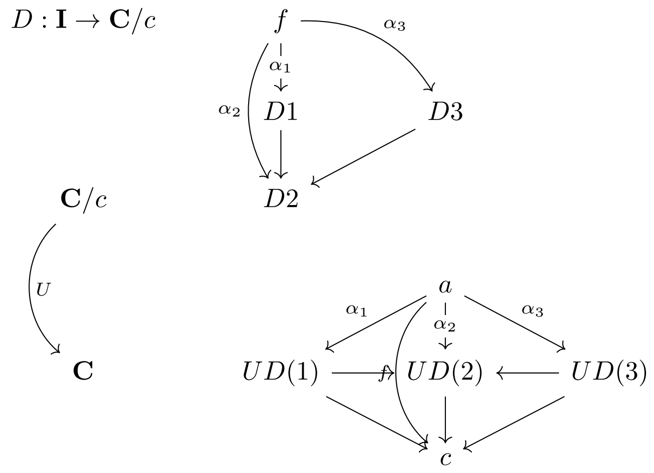
By the naturality of η, any induced morphism $a \to c$ is independent of the choice of the index i, since

$$f_i \circ \eta_i = \eta_* = f_j \circ \eta_j \text{ for all } i, j \in I.$$

The index set I is non-empty. Define a morphism $f : a \to c$, where $f = f_i \circ \eta_i$ for all $i \in I$. By the naturality of η, it follows that $\eta_i : f \to f_i$ is a morphism in $\mathbf{C}/c$ for all $i \in I$ and the naturality holds also in $\mathbf{C}/c$. Therefore η defines a cone over $D : I \to \mathbf{C}/c$. Given any cone $(g : x \to c, \lambda)$ over D in $\mathbf{C}/c$, we have a cone (x, λ) over the diagram D in $\mathbf{C}$. Since the cone morphisms in these two different perspectives $\mathbf{J} \to \mathbf{C}$ and $I \to \mathbf{C}/c$ agree, the unique cone morphism from $(x, \lambda) \to (a, \eta)$ is also the unique cone morphism $(g, \lambda) \to (f, \eta)$. Thus we have shown that products exist in the slice category.

To see the existence of limits, it suffices to show that equalizers exists. To demonstrate the existence of equalizers, we are going to prove a stronger claim. Let $\mathbf{I}$ be a non-empty connected index category, meaning that for every pair of objects a, b in $\mathbf{I}$ there exists a finite sequence of objects $x_1, \ldots, x_n$ in $\mathbf{I}$, where $x_1 = a$ and $x_n = b$ and the set $\mathrm{Hom}(x_i, x_{i+1}) \cup \mathrm{Hom}(x_{i+1}, x_i)$ is non-empty for all $i < n$. Denote the forgetful functor by $U : \mathbf{C}/c \to \mathbf{C}$, where an object is taken to its domain and a morphism is taken to itself in $\mathbf{C}$. Then the functor U strictly creates all limits of the shape $\mathbf{I}$. Since $\mathbf{C}$ is complete, this shows that $\mathbf{C}/c$ has limits of the shape $\mathbf{I}$ and Especially, equalizers. To this end fix a diagram $D : \mathbf{I} \to \mathbf{C}/c$. Let (a, η) be a limit cone over UD in $\mathbf{C}$. We need to show that there exists a unique cone (f, θ) that U takes to (a, η) and moreover (f, θ) is a limit cone over D.

Uniqueness: If U takes the cones (f, α) and (g, β) over D to the same cone (a, λ), then since the forgetful functor U doesn't lose information about the morphisms α_i and β_i in $\mathbf{C}/c$ other than the domain and codomain morphisms, it follows that $\alpha_i = \beta_i$ in $\mathbf{C}$ for all objects i in $\mathbf{I}$. Since $\mathbf{I}$ is non-empty, it follows that $f = D(i) \circ \alpha_i = D(i) \circ \beta_i = g$ in $\mathbf{C}$ for any object i in $\mathbf{I}$. Here is an illustrative picture:



Thus the cones (f, α) and (g, β) are the same in the slice category $\mathbf{C}/c$.

Existence: Since UD has a limit cone (a, η), we use it to define our limit object $f : a \to c$ in $\mathbf{C}/c$. Here we use the connectedness of $\mathbf{I}$: Define $f : a \to c$ to be the morphism $D(i) \circ \eta_i$ for any object i in $\mathbf{I}$. To see that f is independent of the choice of objects, fix objects x and y in $\mathbf{I}$. Now there exists a sequence $x_1, \ldots, x_n$ where $x_1 = x$ and $x_n = y$ and where the sets $\mathrm{Hom}(x_i, x_{i+1}) \cup \mathrm{Hom}(x_{i+1}, x_i)$ are non-empty. Firstly if $n = 2$ and $k : x \to y$, the diagram

$$\begin{tikzcd}
a \arrow[d, "\eta_x"] \arrow[dr, "\eta_y"] & \\
UD(x) \arrow[r, "UD(k)"] \arrow[d, "D(x)"] & UD(y) \arrow[dl, "D(y)"] \\
c &
\end{tikzcd}$$

commutes. The commutativity holds, because η defines a cone over UD and the bottom triangle commutes since $D(k_1)$ is a morphism in the slice category. The case is similar, if there exists $k : y \to x$. Assume that the claim holds if the sequence of objects is of length $m = n - 1$. There exists exists a morphism $x_m \to x_n$ or a $x_n \to x_m$. Since the cases are similar, we may assume that there is a morphism $k : x_m \to x_n$. Now the diagram

$$\begin{tikzcd}
a \arrow[d, "\eta_{x_2}"] \arrow[dr, "\eta_{x_2}"] \arrow[drrr, "\eta_{x_m}"] \arrow[drrrr, "\eta_{x_{m+1}}"] & & & & \\
UD(x_1) \arrow[r, "D(k_1)"] \arrow[d, "D(x_1)"] & UD(x_2) \arrow[dl, "D(x_2)"] & \ldots & UD(x_m) \arrow[r, "D(k_{x_m})"] \arrow[dlll, "D(x_m)"] & UD(x_{m+1}) \arrow[dllll, "D(x_{m+1})"] \\
c & & & &
\end{tikzcd}$$

commutes, since it is a simple joining of two commutative diagrams. Thus η defines a collection of morphisms $f \to D(i)$ natural in i. Therefore (f, η) is a cone over D. From the natural correspondence of the cones over D and UD, which we have essentially already seen, since the construction of the cone (f, η) didn't require that η is a limit cone, one sees that (f, η) is the terminal cone over D. □

Remark 5.40. Dually, it holds that if $\mathbf{C}$ is cocomplete and c is an object of $\mathbf{C}$, then the coslice category $c/\mathbf{C}$ is cocomplete. This yields for example that the pointed categories $\mathbf{Top}_*$ and $\mathbf{Set}_*$ are cocomplete and we have a direct way to compute connected colimits and coproducts.

Chapter 6

Adjoint functors revisited

We have seen the usefulness of adjoint functors. It is fruitful to combine the earlier results relating to adjoint functors and prove a few existence results for adjoints. In this chapter we follow the treatment of theadjoint functor theorems proven in Riehl's book "Category Theory in Context"[6]. Additionally, we will show that every continuous concrete functor that preserves embeddings from an algebraic category to any category $\mathbf{Model}_L^T$ has a left adjoint. Especially all concrete functors, with a codomain whose alphabet doesn't contain relation symbols, between algebraic categories have a left adjoint.

6.1 Existence of adjoints

Theorem 6.1. *Let $F : \boldsymbol{C} \rightleftarrows \boldsymbol{D} : G$ be functors between locally small categories. Then the following are equivalent*

1. *The functor pair (F, G) is an adjoint pair. In other words, there exists a natural transformation $\lambda : I \Rightarrow GF : \boldsymbol{C} \to \boldsymbol{C}$, called a unit, where $(c, \lambda_c : c \to GFc)$ is the initial object in $c \downarrow G$ for all objects c in $\boldsymbol{C}$.*

2. *There exists a natural transformation $\epsilon : FG \Rightarrow I : \boldsymbol{D} \to \boldsymbol{D}$, called a counit, where $(d, \epsilon_d : FGd \to d)$ is terminal in $F \downarrow d$ for all objects d in $\boldsymbol{D}$.*

3. *There exists an isomorphism $\eta_{(c,d)} : \boldsymbol{D}(Fc, d) \cong \boldsymbol{C}(c, Gd)$ natural in objects c and d.*

4. *There exist natural transformations $\lambda : I \Rightarrow GF : \boldsymbol{C} \to \boldsymbol{C}$ and $\epsilon : FG \Rightarrow I : \boldsymbol{D} \to \boldsymbol{D}$, where the unit-counit equations*
$$\begin{cases} \epsilon F \bullet F\lambda = F \\ G\epsilon \bullet \lambda G = G \end{cases}$$
hold.

The correspondences among λ, ϵ and η are canonical: The morphism λ_c is the universal element, the representation of functor $\boldsymbol{C}_c \circ G$, via the natural isomorphism $\eta_{(c,-)}$. Similarly the morphism ϵ_d is the universal element defined by the natural isomorphism $\eta^{-1}_{(-,d)}$, via Yoneda lemma. These two correspondences uniquely define all three natural transformations from any given one.

Proof. The proof is a combination of Adjoint Creation Lemma 3.39 and Corollary 4.12. □

Adjoint Creation Lemma 3.39 states that the mere existence of initial objects in categories $c \downarrow G$, c object in $\mathbf{C}$, implies the existence of a left adjoint F for G. Furthermore the initial objects of $c \downarrow G$, for objects c in $\mathbf{C}$, defines the unit of $F \dashv G$.

6.1.1 General Adjoint Functor Theorem

Definition 6.2. Let $\mathbf{C}$ be a category and let S be a small set of objects of $\mathbf{C}$. We say that S is a jointly weakly initial set, if for any object c in $\mathbf{C}$ there exists a morphism $s \to c$ for some object $s \in S$. An object is called weak initial object, if its singleton is jointly weakly initial.

If a category has products and a jointly weakly initial set of objects S, then it has a weakly initial object by taking the product of objects in S.

Theorem 6.3. *Let $\mathbf{C}$ be a locally small complete category with a jointly weakly initial set S. Then $\mathbf{C}$ has an initial object.*

Proof. Let c be the product object of the objects in S. Now c is weakly initial. The inclusion $D : \mathbf{C}(c,c) \hookrightarrow \mathbf{C}$, where $\mathbf{C}(c,c)$ is thought as a one object category, is a diagram in $\mathbf{C}$. Since $\mathbf{C}$ is complete, there exists a limit cone (a, i) over D, where $i : a \to c$. Notice that $fi = i$ for all morphisms $f : c \to c$ in $\mathbf{C}$ and i is monic by Theorem 5.9. Now a is a weakly initial object. We will show that a is an initial object in $\mathbf{C}$. Assume that $f, g : a \to x$ are morphisms in $\mathbf{C}$. We will show that they must be the same. Denote the equalizer of f and g by $k : b \to a$. It suffices to show that k is a retraction and hence epic, since $fk = gk$. There exists a morphism $h : c \to b$ by the weak initiality of c:

$$\begin{tikzcd} b \arrow[r, "k"] & a \arrow[r, "i"] & c \arrow[ll, bend left, "h"] \end{tikzcd}$$

Consider the endomorphism $i \circ k \circ h$ on c. Now $i \circ k \circ h \circ i = i$. By monicness of i, $khi = id$. Thus k is a retraction. Hence $f = g$. □

Corollary 6.4 (General Adjoint Functor Theorem)**.** *Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Assume that $\mathbf{D}$ is a complete locally small category. Let $G : \mathbf{D} \to \mathbf{C}$ be a continuous functor. Assume that for every object c in $\mathbf{C}$ there exists a jointly weakly initial set S_c in $c \downarrow G$. Then G has a left adjoint.*

Proof. Since G is continuous and $\mathbf{D}$ is complete, it follows that $c \downarrow G$ is a complete category by Theorem 5.5. Because $c \downarrow G$ is complete, locally small and has a jointly weakly initial set S_c, it follows that $c \downarrow G$ has an initial object for all objects c in $\mathbf{C}$. Whence G has a left adjoint. □

We say that a functor $G : \mathbf{D} \to \mathbf{C}$ satisfies the solution set condition if for every object c in $\mathbf{C}$ there exists a weakly initial set of objects in $c \downarrow \mathbf{C}$. The General Adjoint Functor Theorem yields many interesting left adjoints.

Theorem 6.5. *Let L_i be an alphabet and let T_i be an L_i-theory for $i = 1, 2$. Assume that T_1 is an algebraic theory. Let $U_i : \mathbf{Model}_{L_i}^{T_i} \to \mathbf{Set}$ be the forgetful functor for $i = 1, 2$. Assume that $G : \mathbf{Model}_{L_1}^{T_1} \to \mathbf{Model}_{L_2}^{T_2}$ is a continuous concrete functor. Assume that G preserves embeddings. Then G is a right adjoint functor.*

Proof. Since G is continuous and the category $\mathbf{Model}_{L_1}^{T_1}$ is complete and locally small, by Theorem 5.31 it suffices to show that G satisfies the solution set condition. Let $\mathcal{M}$ be an L_2, T_2-model, with a universe M. We find a jointly weakly initial set of objects of $\mathcal{M} \downarrow G$: Consider the possibly large set $\mathcal{H}$ of all L_1-models $\mathcal{N}$ that are generated by a subset of cardinality at most $\mathrm{Card}(M)$. Choose the set of representatives $\mathcal{A}$ of the isomorphism classes of models in $\mathcal{H}$.

The set $\mathcal{A}$ is small, since the cardinality of models in $\mathcal{A}$ is bounded. The boundedness follows from the fact that T_1 is an algebraic theory and hence every subset of the universe of an L_1, T_1-model $\mathcal{N}$ generates a L_1, T_1-submodel by adding the interpretations of constant symbols and applying the functions $f^{\mathcal{N}}$ recursively countably many times for L_1-function symbols f. Define S to be the set of all L_2-model morphisms $\mathcal{M} \to G(\mathcal{N})$, where $\mathcal{N} \in \mathcal{A}$.

Let $\mathcal{N}$ be an L_1-model, with a universe N, that satisfies the theory T_1 and let $f : \mathcal{M} \to G(\mathcal{N})$ be an L_2-model morphism. The image of the function $U_2(f) = f : M \to N$ with the set $\{c^{\mathcal{N}} \mid c \text{ a constant symbol of } L_1\}$ generates recursively a set N' via the closure operations $g^{\mathcal{N}}$ for function symbols g of L_1. The set N' defines a full submodel $\mathcal{N}'$ of $\mathcal{N}$. Since the theory T_1 is algebraic and the inclusion $\mathcal{N}' \hookrightarrow \mathcal{N}$ a globally full injection, it follows that $\mathcal{N}'$ satisfies the theory T_1 by 1.41(3). Since the inclusion $\mathcal{N}' \overset{i}{\hookrightarrow} \mathcal{N}$ is an embedding by Example 2.43, it follows, by assumption, that $G(\mathcal{N}') \xrightarrow{G(i)} G(\mathcal{N})$ is an embedding. Consider the commutative diagram

$$\begin{tikzcd} M \arrow[r, "f|"] \arrow[dr, "f"'] & N' \arrow[d, dashed, "G(i)"] \\ & N \end{tikzcd}$$

of functions. Since $G(i)$ is an embedding and f is a L_2-model morphism, it follows that $f| : \mathcal{M} \to G(\mathcal{N}')$ is an L_2-model morphism. By definition of $\mathcal{A}$ there exists a model $\mathcal{N}'' \in \mathcal{A}$ and an isomorphism $k : \mathcal{N}'' \cong \mathcal{N}'$. Hence we attain the commutative diagram

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f|"'] \arrow[d, dashed, "f'"] \arrow[rr, bend left, "f"] & G(\mathcal{N}') \arrow[r, "G(i)"'] & G(\mathcal{N}) \\
G(\mathcal{N}'') \arrow[ur, "G(k)"'] & &
\end{tikzcd}$$

Thus the set S is jointly weakly initial. Therefore G has a left adjoint functor. □

Corollary 6.6. *Let L_i be an alphabet and let T_i be an algebraic L_i-theory for $i = 1, 2$. Assume that L_2 contains no relation symbols. Let $G : \boldsymbol{Model}_{L_1}^{T_1} \to \boldsymbol{Model}_{L_2}^{T_2}$ be a concrete functor. Then G has a left adjoint. Especially the forgetful functor $U_i : \boldsymbol{Model}_{L_i}^{T_i} \to \boldsymbol{Set}$ has a left adjoint for $i = 1, 2$.*

Proof. By the construction of equalizer submodel and product models, we see that the functors U_1 and U_2 preserve products and equalizers. Thus by Theorem 5.30 the functors U_1 and U_2 are continuous. Since the alphabet L_2 contains no relation symbols, it follows that U_2 reflects isomorphisms by Theorem 1.17. Since the domain of U_2 complete, U_2 is continuous and U_2 reflects isomorphisms, it follows that U_2 reflects limits by 5.24. Since G is a concrete functor, it holds that $U_1 = U_2 G$. Since U_1 preserves limits and U_2 reflects limits, it directly follows that G is continuous. Furthermore all injection in $\mathbf{Model}_{L_2}^{T^2}$ are embeddings. From Theorem 6.5 it follows that G is a right adjoint. □

In a very similar manner as in the proof of Theorem 6.5 we are able to prove the existence of the tensor product:

Theorem 6.7 (Existence of the tensor product)**.** *Let L_i be an alphabet and let T_i be an algebraic L_i-theory for $i = 1, 2$. Assume that the alphabet L_2 contains no relation symbols and assume that $F : \boldsymbol{Model}_{L_1}^{T_1} \to \boldsymbol{Model}_{L_2}^{T^2}$ is a concrete functor. Denote the forgetful functor by $U_i : \boldsymbol{Model}_{L_i}^{T_i} \to \boldsymbol{Set}$ for $i = 1, 2$ and fix an L_1, T_1-model $\mathcal{M}$. Assume that the set $Hom(\mathcal{M}, \mathcal{N})$ defines an L_2-submodel of*

$$F(\prod_{x \in U_1(\mathcal{M})} \mathcal{N})$$

for all L, T-models $\mathcal{N}$. Then the covariant hom-functor $G := Hom_{\mathcal{M}} : \boldsymbol{Model}_{L_1}^{T_1} \to \boldsymbol{Model}_{L_2}^{T_2}$ has a left adjoint.

Proof. Since the theory T_2 is algebraic, it follows that $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an L_2, T_2-model by Theorem 1.41(3). Let $f : \mathcal{N} \to \mathcal{N}'$ be a L-model morphism. The post composition map $G(f)$ is an L_2-model morphism and therefore G is well defined as a functor. The continuity of the functor G follows from the fact that the hom-functor $U_2 G$ preserves limits (Theorem 5.25) and U_2 reflects limits.

The functor G satisfies the solution set condition by a similar argument as given in the proof of Theorem 6.5. □

Example 6.8.

1. The forgetful functors from **Mon**, **Grp**, **R-Mod** to **Set** have a left adjoint.

2. The forgetful functor from R-modules to groups has a left adjoint functor.

3. The category of abelian monoids has a tensor product: The hom-functor $\mathrm{Hom}_M : \mathbf{Mon} \to \mathbf{Mon}$ has a left adjoint for every monoid M.

6.1.2 Special Adjoint Functor Theorem

In the proof of the General Adjoint Functor Theorem it was essential to be able to use big equalizers to generate the smallest possible subobject which then happened to be the initial object. We are now going to present a familiar but still a different method of generating an initial object.

Definition 6.9. A set S of objects of a category $\mathbf{C}$ is called a separating set if the collection of hom-functors $\mathbf{C}_a, a \in S$, is jointly faithful, meaning that if $\mathbf{C}_a(f) = \mathbf{C}_a(g)$ for all objects $a \in S$, then $f = g$, for any parallel morphisms f and g in $\mathbf{C}$. Dually, a coseparating set S in a category $\mathbf{C}$ is such that the functors $\mathbf{C}^a, a \in S$ are jointly faithful.

Intuitively, a separating set of objects in a category $\mathbf{C}$ refers to a collection of shapes that test how similar two morphisms are. For example, in the category $\mathbf{Set}$ of sets the terminal set 1 separates all morphisms. We see that $f = g$ if and only if they map the generalized elements of shape 1 similarly. Put differently, $f = g$ if and only if $\mathbf{Set}_1(f) = \mathbf{Set}_1(g)$. The set $1 \sqcup 1 =: 2$ works as a coseparating object, since to check that two parallel functions $f, g : X \to Y$ agree on a point x, it suffices to look at the characteristic function $\tau : Y \to 2$ of the set $\{f(x)\}$. Now if $\tau \circ f = \tau \circ g$, then $f(x) = g(x)$.

In topology, Urysohn's lemma says that every two disjoint closed sets A and B in a normal topological space[1] X are witnessed to be disjoint by a continuous function $f : X \to [0, 1]$, whereby $A \subset f^{-1}\{0\}$ and $B \subset f^{-1}\{1\}$. Hence in the category of normal Hausdorff spaces, or more specifically compact Hausdorff spaces, the interval $[0, 1]$ works as a coseparating object.

Lemma 6.10. *Assume that $\mathbf{C}$ is a locally small and complete category with a small coseparating set S of objects and assume the intersection[2] of subobjects of any fixed object exists. Then $\mathbf{C}$ has an initial object.*

Proof. Denote the product object of the objects in S by t. The intersection of subobjects of t exists, denote it by $i : c \to t$. We will show that c is an initial object in $\mathbf{C}$. Given an object a in $\mathbf{C}$, we need to show that there exists a unique morphism $c \xrightarrow{!} a$. Uniqueness is easily seen by the fact that if there exist two morphisms $\alpha, \beta : c \to a$, then their equalizer would be a subobject of c and by composing with i, a subobject of t. By the minimality of i we see that the equalizer of α and β is an isomorphism and hence $\alpha = \beta$.

It remains to show the existence: To construct the morphism $c \to a$, we are going to use a pullback to construct a subobject for t so that there exists a morphism from the subobject to a. Using the universal property of products there exists a canonical morphism $a \to \prod_{j \in \mathbf{C}(a,s)} s$ for any object s in S. Using again the universal property of product, we obtain the canonical morphism

$$a \xrightarrow{f} \prod_{s \in S} \Big(\prod_{j \in \mathbf{C}(a,s)} s \Big).$$

The fact that S is a coseparating set of objects is equivalent to f being monic. Consider a parallel pair of morphisms x and x' with codomain a. Now $fx = fx'$ is equivalent to $f'x = f'x'$ for all $f' : a \to s$ and $s \in S$.

Additionally, we have the morphisms $s \to \prod_{j \in \mathbf{C}(a,s)} s$ for $s \in S$, where composing with projections yields identities. Taking the product morphism

$$\prod_{s \in S} s \to \prod_{s \in S} \Big(\prod_{j \in \mathbf{C}(a,s)} s \Big),$$

we are ready for a pullback diagram

$$\begin{tikzcd}
x \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, very near start, "\lrcorner"] & a \arrow[d, "f"] \\
\prod_{s \in S} s \arrow[r] & \prod_{s \in S} (\prod_{j \in \mathbf{C}(a,s)} s)
\end{tikzcd}$$

for some object x and some morphism f' in $\mathbf{C}$. Since f is monic, so is f' and hence f' defines a subobject of $t = \prod_{s \in S} s$. Since $i : c \to t$ is the smallest subobject of t, we have a morphism $c \to x$ and hence a morphism $c \to a$. $\square$

[1] A space is called normal, if every two disjoint closed sets can be separated by disjoint open neighbourhoods, respectively.

[2] The intersection of subobjects means the generalized pullback of the monomorphisms, in other words a product in the slice category.

Theorem 6.11 (Special Adjoint Functor Theorem)**.** *Let $\boldsymbol{C}$ and $\boldsymbol{D}$ be locally small categories, where $\boldsymbol{D}$ is complete and evey subclass of subobjects of any fixed object has an intersection. Assume that $\boldsymbol{D}$ has a coseparating set of objects S. Then G is a right adjoint functor.*

Proof. Fix an object c in $\mathbf{C}$. We show that $c \downarrow G$ has an initial object by applying Lemma 6.10. For this we show that $c \downarrow G$ is locally small, complete and every object has a minimal subobject. By Theorem 5.37, $c \downarrow G$ is complete and clearly it is locally small. Consider an object (d, f) of $c \downarrow G$ and a possibly large diagram D that chooses a monic from every subobject of (d, f). The forgetful functor $R : c \downarrow G \to \mathbf{D}$ creates and preserves limits by Theorem 5.5. Thus R maps subobjects of (d, f) injectively to subobjects of d. By assumption the diagram RD has a limit that defines a subobject of d. Even though the D might be a large diagram, R creates the limit. This follows from Theorem 5.36 and the fact that any large diagram becomes a diagram as the universe is enlarged suitably. Therefore D has a limit that defines the smallest subobject of (d, f).

It remains to define a coseparating set of objects for $c \downarrow G$. Define the set T to consist of the objects (s, f) in $c \downarrow G$ where $s \in S$. Since S is a small set and $\mathbf{C}$ locally small, T is a small set. The set T is a coseparating set: Let $\alpha, \beta : (d, f) \to (d', f')$ be morphisms in $c \downarrow G$. Assume that $\theta\alpha = \theta\beta$ for all $\theta : (d', f') \to t$ and $t \in T$.

We need to show that $\alpha = \beta$. Let $\theta : d' \to s$ be a morphism in $\mathbf{D}$, where $s \in S$. It suffices to show that $\theta\alpha = \theta\beta$ by the coseparating property of S. Now θ becomes a morphism $(d', f') \to (s, G(\theta)f')$ in $c \downarrow G$. Because $(s, G(\theta)f') \in T$, from the assumption if follows that $\theta\alpha = \theta\beta$ in $c \downarrow G$ and especially $\theta\alpha = \theta\beta$ in $\mathbf{C}$. Hence $\alpha = \beta$. □

Remark 6.12. The proof of Theorem 6.11 generalizes the Stone–Čech compactification and since the proof is constructive, we can find the compactification: The category of compact Hausdorff spaces satisfies completeness (Tychonoff's Theorem), local smallness and any set of compact subsets has a compact intersection. Furthermore the unit interval defines a coseparating object. Therefore the forgetful functor from the category of compact Hausdorff spaces **CHTop** to **Top** has a left adjoint β, since G is continuous. Since the proof of Special Adjoint Functor Theorem is constructive, we can explicitly construct the compactification functor $\beta : \mathbf{Top} \to \mathbf{CHTop}$:

Consider the proof of Theorem 6.11 and denote the unit interval $[0, 1]$ by I. In the case of the forgetful functor $G : \mathbf{CHTop} \hookrightarrow \mathbf{Top}$ fix the left adjoint β of G and fix a space X. The proof constructs a component $\eta_X : X \to G\beta X$ of a unit at the same time as the space βX. The set T consists of all continuous maps $X \to G[0, 1]$ and T defines a coseparating set in $X \downarrow G$. Lemma 6.10 constructs the initial object of $X \downarrow G$ from T by taking the smallest subobject of the product of objects in T. The product of objects in T is the canonical map $X \to G(I^{Hom(X,I)})$, which is seen in the proof of Theorem 5.36. The smallest subobject of the map $X \to GI^{Hom(X,I)}$ is the topological closure of the set theoretical image. Thus the left adjoint β takes the space X to the closure of the image of the canonical map $X \xrightarrow{\lambda_X} I^{Hom(X,I)}$ and the unit is the corestriction $X \xrightarrow{\lambda_X|} \overline{\mathrm{im}(\lambda_X)}$. From this information the functor β is uniquely defined by Adjoint Creation Lemma 3.39.

Definition 6.13. Let $\mathbf{C}$ be a category. We say that $\mathbf{C}$ is well powered if the collection of subobjects $\mathrm{Sub}(c)$ of c is a small set for every object c in $\mathbf{C}$.

If a category $\mathbf{C}$ is well powered and complete, then for every object c of $\mathbf{C}$, there exists a diagram D that chooses a monic from every subobject of c. The limit of D then defines the smallest subobject of c. Therefore we have a corollary:

Corollary 6.14. *Let $G : \boldsymbol{D} \to \boldsymbol{C}$ be a continuous functor between locally small categories. Assume that $\boldsymbol{D}$ is complete and well powered with a small set of coseparating objects. Then G is a right adjoint functor.*

From this corollary we see that any increasing map $f : P \to Q$ between prosets, where P is complete, is a right Galois connection if and only if f preserves infimums.

The Special Adjoint Functor Theorem has a surprising corollary.

Corollary 6.15. *Let $\boldsymbol{C}$ be a locally small and complete category with a small coseparating set of objects. Assume that the intersection of any collection of subobjects of any fixed object exists in $\boldsymbol{C}$ Then $\boldsymbol{C}$ is cocomplete.*

Proof. Fix a small category $\mathbf{I}$. It suffices to show that the constant embedding functor $\Delta : \mathbf{C} \to [\mathbf{I}, \mathbf{C}]$ has a left adjoint. The functor category $[\mathbf{I}, \mathbf{C}]$ is locally small and the category $\mathbf{C}$ satisfies the assumptions of the Special Adjoint Functor Theorem. It follows that Δ has a left adjoint. This shows that the colimits of shape $\mathbf{I}$ exists in $\mathbf{C}$. $\square$

It follows that the category **CHTop** of compact Hausdorff spaces is a cocomplete category. The infinite coproduct of **CHTop** spaces differs from the corresponding coproduct in the category of topological spaces.

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